Structured matrices, continued fractions, and root localization of polynomials

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Abstract

We give a detailed account of various connections between several classes of objects: Hankel, Hurwitz, Toeplitz, Vandermonde and other structured matrices, Stietjes and Jacobi-type continued fractions, Cauchy indices, moment problems, total positivity, and root localization of univariate polynomials. Along with a survey of many classical facts, we provide a number of new results.

Contents

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Introduction

This survey and research paper offers a glimpse at several classical topics going back to Descartes, Gauss, Stieltjes, Hermite, Hurwitz and Sylvester (see, e.g., [19, 80, 28, 81, 82]), all connected by the idea that behavior of polynomials can be analyzed via algebraic constructs involving their coefficients. Thus a number of inter-related algebraic constructs was built, including Hurwitz, Toeplitz and Hankel matrices, the corresponding quadratic forms, and the corresponding continued fractions. The linear-algebraic properties of these objects were shown to be intimately related to root localization of polynomials such as stability, whereby the zeros of a polynomial avoid a specific half-plane, or hyperbolicity, whereby the zeros lie on a specific line. These methods gave rise to several well-known tests of root localization, such as the Routh-Hurwitz algorithm or the Lienard-Chipart test.

In the 20th century, this line of research was developed further by Schur [24], Pólya [78], Krein [59, 60, 34] and others (see, e.g., [43]), leading to important notions of total positivity, Pólya frequency sequences, stability preservers etc. A very important part of that research effort was devoted to entire functions, in particular, the Laguerre-Pólya and related classes. However, this area gradually went out of fashion and was essentially abandoned around 1970-1980s, with a few exceptions, such as Padé approximation. We should note in this connection that the closely related theory of continued fractions, initiated by Chebyshev, Stieltjes and Markov and further developed by Akhiezer, Krein and their collaborators in connection with problems of mechanics is only now returning to the forefront due to its connections with orthogonal polynomials [58, 66, 69, 49, 50].

On the other hand, various fast algorithms for structured matrices (Cauchy, Vandermonde, Toeplitz, Hankel matrices, and more generally quasiseparable matrices and matrices with displacement structure) have been developing rapidly in the last few decades due to the efforts of Gohberg, Olshevsky, Eidelman, Kailath, Heinig, Sakhnovich, Lerer, Rodman, Fuhrmann and others, along with applications to control theory and engineeting: see the collections [70, 71, 72, 53, 15, 22] and the references therein. Surveying these developments is, however, outside the scope of this paper.

Furthermore, we must stress the fact that the results and algorithms in this paper are not approximate but exact (i.e., give exact answers in exact arithmetic) as is the term *localization*, which is often used in the sequel. For instance, we perform root localization with respect to a line or a half plane and are able to determine *exactly*, e.g., how many zeros of a given polynomial lie on a given line. For sure, this does not obviate the need for error analysis of the corresponding algorithms that use finite precision or floating-point arithmetic.

We must note that the questions of root localization are returning into the spotlight also due to the development of basic algebraic techniques for multivariate polynomials [14, 40, 18, 17, 16, 83] and entire functions [9, 20], as well as due to newly found applications of results on multivariate stable and hyperbolic polynomials to other branches of mathematics [42, 41, 100, 16]. Very promising applications of current interest include problems of discrete probability, combinatorics, and statistical physics, such as the analysis of partition functions arising in classical Ising and Potts models of statistical mechanics [92, 89].

The goal of this paper is to provide a comprehensive and coherent treatment of classical connections between three kinds of objects: various structured matrices, representations of rational functions via continued fractions, and root localization of univariate polynomials. Our additional goal is to demonstrate that, despite the rich history of this subject, even the univariate case is far from being exhausted, and that classical algebraic techniques are useful in answering questions about the behavior of polynomials and rational functions.

Our future goals include using this work for generalizations of these classical results to univariate entire and meromorphic functions, including questions on Pólya frequency sequences, Hurwitz rational functions, and entire functions with all real zeros [98, 12]. One particular area of interest is the class of so-called generalized Hurwitz polynomials, which is a useful large class containing all Hurwitz stable polynomials. This line of our ongoing research is closely related to the theory of indefinite metric spaces, indefinite moment problems, and the corresponding eigenvalue problems [23, 76, 77, 47].

We now illustrate our main point above by discussing several new results in this paper.

Our first illustration is provided by the body of work in Section 4.2. These results provide explicit criteria for a polynomial to have only real roots in terms of Hankel and Hurwitz determinants made of its coefficients. To give the reader an idea of these statements, we quote two sample theorems from Section 4.2:

Consider a polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad a_1, \ldots, a_n \in \mathbb{R},\ a_0 > 0.$$

and let

$$L(z) := \frac{p'(z)}{p(z)} = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots$$

be its logarithmic derivative. Define the associated infinite Hankel matrix

$$S := S(L) := \|s_{i+j}\|_{i,j=0}^{\infty}$$

and its leading principal minors

$$D_j(S) := D_j(S(L)) := \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{j-1} \\ s_1 & s_2 & s_3 & \ldots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \ldots & s_{2j-2} \end{vmatrix}, \quad j = 1, 2, 3, \ldots.$$

Here is a sample result from Section 4.2 that provides an interesting connection between total positivity of a special Hurwitz matrix and polynomials all whose roots are real and negative:

Theorem (Total positivity criterion for negative zeros). *The polynomial*

$$g(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_0 \neq 0, \quad a_n > 0$$

has all negative zeros if and only if the infinite matrix

$$\mathcal{D}_\infty(g) := \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\ 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 & 6a_6 & \ldots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\ 0 & 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 & \ldots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ 0 & 0 & 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative.

This result is based on a new connection between Stieltjes continued fractions representing a rational function $P = q/p$ and a special factorization of the infinite Hurwitz matrix $H(p,q)$ associated to the pair (p,q) (see Section 3.4). This connection implies a criterion of total nonnegativity of the latter infinite Hurwitz matrix $H(p,q)$ (Theorem 3.44), which in turn implies our criterion for negative zeros of a polynomial g via the total nonnegativity of the infinite discriminant matrix $\mathcal{D}_\infty(g)$. One direction of Theorem 3.44 was essentially developed in the earlier works of Asner and Kemperman[6, 57], while the other direction is given here for the first time. This result simultaneously provides a criterion of stability [6, 57, 46, 55, 98].

Section 4.2 provides many other alternative criteria for real zeros, including the cases of only positive, only negative, or mixed zeros, in terms of Hankel and Hurwitz determinants as well as the coefficients of Stieltjes continued fraction expansion of the logarithmic derivative of a given polynomial. The special role played by the logarithmic derivative for the analysis of real zeros is clarified earlier in Section 4, together with the main ideas behind counting real roots to the left and to the right of the origin. The underlying theory of Cauchy indices is presented in Section 3.3.

Another important class of objects behind the results of this paper on polynomial roots is the class of so-called R-functions. These are rational functions mapping the upper-half plane of the complex plane either to the lower half-plane or to itself (see Definition 3.1). The theory of R-functions connects several key objects of this work: continued fractions, Cauchy indices, polynomials with real roots, the moment problem, and determinantal inequalities. As an example, we quote the following new result from Section 3.3:

Theorem (Generalized Lienard-Chipart criterion). *If a real rational function $R = q/p$, with p and q relatively prime, maps the upper half-plane of the complex plane to the lower half-plane, then the number of*

its positive poles is equal to the number $\mathrm{V}^-(a_0,\dots,a_n)$ *of strong sign changes in the sequence* $(a_0,\dots,a_n)$ *of coefficients of its denominator* p*. In particular,* R *has only negative poles if and only if* $a_j>0$ *for* $j=1,2,\dots,n$*, and only positive poles if and only if* $a_{j-1}a_j<0$ *for* $j=1,2,\dots,n$*.*

Our final illustration is the generalized Orlando's formula from Section 1.2. Orlando's formula *per se* expresses the Hurwitz determinant of order $n-1$ associated to a polynomial p of degree n as the product, up to a certain normalization, of all possible pairs z_i+z_j of the zeros of p. Orlando's result goes back to 1911 [73]. The generalized Orlando formula obtained in this paper says the following:

Theorem (Generalized Orlando formula). *Let* p *be a polynomial of degree* n *and* q *a polynomial of degree* $m\leq n$*. Then the resultant of these polynomials can be computed as follows:*

$$\mathbf{R}(p,q)=(-1)^{\frac{n(n-1)}{2}}a_0^{m+n}\prod_{1\leq i<j\leq 2n}(z_i+z_j)$$

where a_0 *is the leading coefficient of* p *and where* z_i*,* $i=1,\dots,2n$*, are the zeros of the polynomial*

$$h(z):=p(z^2)+zq(z^2).$$

The classical Orlando's formula then follows as a special case by splitting an arbitrary polynomial into its even and odd part and applying the generalized Orlando's formula.

As these examples show, new connections can be found between several classical matrix classes (Hurwitz, Hankel, Vandermonde etc.) made of the coefficients of polynomials, different representations of rational functions (Stieltjes and Jacobi continued fractions, Laurent series, elementary fractions), and various counting notions (sign changes, the number of roots in a specific domain of a complex plain, Cauchy indices). Underlying all these is a coherent theory demonstrating how these connections arise.

The presented theory is completely general in that it does not single out the special case of stable polynomials. The importance of stability, historically the first question on root localization [61, 51, 32, 84, 37], is by now quite well understood. Various stability criteria have been studied in great detail, especially in the engineering literature. In this survey (Section 4), we choose instead to illustrate applications of the general theory to polynomials with real roots, a less studied but arguably equally important special case.

To summarize, the point of this work is to provide a uniform, streamlined, algebraic treatment for a body of questions centered around the root localization of polynomials. At present, results of this type are scattered in the literature, whereas even textbooks and monographs about polynomials sometimes fail to provide answers to some basic questions (i.e., tests for real roots using only the coefficients of the polynomial). We hope that this work will serve as a useful reference and source of further research for mathematicians in various fields interested in polynomials and their zeros.

1 Complex rational functions and related topics

Consider a rational function

$$z\mapsto R(z):=\frac{q(z)}{p(z)} \tag{1.1}$$

where p and q are polynomials with complex coefficients

$$p(z) := a_0z^n+a_1z^{n-1}+\cdots+a_n, \qquad a_0,\,a_1,\,\dots,\,a_n\in\mathbb{C},\ a_0\neq 0, \tag{1.2}$$

$$q(z) := b_0z^n+b_1z^{n-1}+\cdots+b_n, \qquad b_0,\,b_1,\,\dots,\,b_n\in\mathbb{C}, \tag{1.3}$$

If the greatest common divisor of p and q has degree l, $0\leq l\leq n$, then the rational function R has exactly $r=n-l$ poles. Note that zeros or poles of a rational function are always counted with multiplicities unless explicitly stipulated otherwise. Thus, in the rest of the paper, the phrase "counting multiplicities" will be implicit in every statement about zeros or poles of functions under consideration.

1.1 Hankel and Hurwitz matrices. Hurwitz formulæ

In this section, we introduce Hankel and Hurwitz matrices associated to a given rational functions and discuss the Hurwitz formulæ that connect these two classes.

Expand the function (1.1) into its Laurent series at ∞:

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots . \tag{1.4}$$

Here $s_j = 0$ for $j < n-1-m$ and $s_{n-1-m} = \dfrac{b_0}{a_0}$, where $m = \deg q$.

The sequence of coefficients of negative powers of z

$$s_0, s_1, s_2, \ldots$$

defines the infinite *Hankel* matrix $S := S(R) := \|s_{i+j}\|_{i,j=0}^{\infty}$. This gives the correspondence

$$R \mapsto S(R). \tag{1.5}$$

Definition 1.1. For a given infinite sequence $(s_j)_{j=0}^{\infty}$, consider the determinants

$$D_j(S) := \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{j-1} \\ s_1 & s_2 & s_3 & \dots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \dots & s_{2j-2} \end{vmatrix}, \quad j = 1, 2, 3, \ldots, \tag{1.6}$$

i.e., the *leading principal minors* of the infinite Hankel matrix S. These determinants are referred to as the *Hankel minors* or *Hankel determinants.*

An infinite matrix is said to have finite rank r if all its minors of order greater than r are zero whereas there exists at least one nonzero minor of order r. Knonecker [63] proved that, for any infinite Hankel matrix, any minor of order r where r is the rank of the matrix, is a multiple of its leading principal minor of order r. This implies the following result.

Theorem 1.2 (Kronecker [63])**.** *An infinite Hankel matrix $S = \|s_{i+j}\|_{i,j=1}^{\infty}$ has finite rank r if and only if*

$$D_r(S) \neq 0, \tag{1.7}$$
$$D_j(S) = 0, \quad \text{for all } j > r. \tag{1.8}$$

Let $\widehat{D}_j(S)$ denote the following determinants

$$\widehat{D}_j(S) := \begin{vmatrix} s_1 & s_2 & s_3 & \dots & s_j \\ s_2 & s_3 & s_4 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_j & s_{j+1} & s_{j+2} & \dots & s_{2j-1} \end{vmatrix}, \quad j = 1, 2, 3, \ldots, \tag{1.9}$$

With a slight abuse of notation, we will also write $D_j(R) := D_j(S(R))$ and $\widehat{D}_j(R) := \widehat{D}_j(S(R))$ if the matrix $S = S(R)$ is made of the coefficients (1.4) of the function R.

The following theorem was also established by Gantmacher in [36].

Theorem 1.3. *An infinite matrix $S = \|s_{i+j}\|_{i,j=1}^{\infty}$ has finite rank if and only if the sum of the series*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots$$

is a rational function of z. In this case the rank of the matrix S is equal to the number of poles of the function R.

Theorems 1.2 and 1.3 have a simple corollary, which will be useful later.

Corollary 1.4. *A rational function R with exactly r poles represented by the series* (1.4) *has a pole at the point* 0 *if and only if*

$$\widehat{D}_{r-1}(R) \neq 0 \quad \textit{and} \quad \widehat{D}_j(R) = 0 \quad \text{for } j = r, r+1, \ldots . \tag{1.10}$$

Proof. Since the function R represented as a series (1.4) has exactly r poles, the function

$$G(z) := zR(z) - zs_{-1} = s_0 + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3} + \cdots$$

has exactly $r-1$ poles if and only if R has a pole at the point 0. If R does not have a pole at 0, then G has r poles. Thus, by Theorems 1.2 and 1.3, the function R has a pole at 0 if and only if

$$D_{r-1}(G) \neq 0 \quad \text{and} \quad D_j(G) = 0, \quad j = r, r+1, \ldots. \tag{1.11}$$

Since

$$\widehat{D}_j(R) = D_j(G), \quad j = 1, 2, \ldots,$$

the formula (1.11) yields the assertion of the corollary. □

Theorems 1.2 and 1.3 imply the following: if the greatest common divisor of the polynomials p and q defined in (1.2)–(1.3) has degree l, $0 \leq l \leq \deg q$, then the formulæ (1.7)–(1.8) hold for $r = n - l$ for the rational function (1.1), where n is the degree of the polynomial (1.2), i.e., the denominator of R.

Denote by $\nabla_{2j}(p,q)$ the following determinants of order $2j$:

$$\nabla_{2j}(p,q) := \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & a_{j-1} & a_j & \ldots & a_{2j-1} \\ b_0 & b_1 & b_2 & \ldots & b_{j-1} & b_j & \ldots & b_{2j-1} \\ 0 & a_0 & a_1 & \ldots & a_{j-2} & a_{j-1} & \ldots & a_{2j-2} \\ 0 & b_0 & b_1 & \ldots & b_{j-2} & b_{j-1} & \ldots & b_{2j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_0 & a_1 & \ldots & a_j \\ 0 & 0 & 0 & \ldots & b_0 & b_1 & \ldots & b_j \end{vmatrix}, \quad j = 1, 2, \ldots, \tag{1.12}$$

constructed using the coefficients of the polynomials (1.2)–(1.3). Here we set $a_i = 0$ for all $i > n$ and $b_l = 0$ for $l > m$. The determinants $\nabla_{2j}(p,q)$ are called *determinants of Hurwitz type* or just *Hurwitz determinants* or *Hurwitz minors*.

In his celebrated work [48], A. Hurwitz found relationships between the minors $D_j(R)$ defined in (1.6) and the determinants $\nabla_{2j}(p,q)$ defined by (1.12).

Theorem 1.5 ([48, 61, 36, 11])**.** *Let* $R(z) = \dfrac{q(z)}{p(z)}$*, where the polynomials* p *and* q *are defined by* (1.2)–(1.3)*. Then*

$$\nabla_{2j}(p,q) = a_0^{2j} D_j(R), \quad j = 1, 2, \ldots. \tag{1.13}$$

Proof. From (1.4) it follows that coefficients s_k, a_k, b_k satisfy recurrence relations

$$b_j = a_0 s_{j-1} + a_1 s_{j-2} + \cdots + a_j s_{-1}, \qquad j = 0, 1, 2, \ldots. \tag{1.14}$$

These recurrence relations imply the formulæ (1.13) by direct matrix multiplication, once we take into

account (1.14) (cf. [61]):

$$
\begin{aligned}
a_0^{2j+1}D_j(R) &= (-1)^{\frac{j(j-1)}{2}} a_0^{2j+1}
\begin{vmatrix}
1 & 0 & 0 & \dots & 0 & 0 & \dots & 0\\
0 & 1 & 0 & \dots & 0 & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & 1 & 0 & \dots & 0\\
0 & s_{-1} & s_0 & \dots & s_{j-2} & s_{j-1} & \dots & s_{2j-2}\\
0 & 0 & s_{-1} & \dots & s_{j-3} & s_{j-2} & \dots & s_{2j-3}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & s_0 & s_1 & \dots & s_{j-2}\\
0 & 0 & 0 & \dots & s_{-1} & s_0 & \dots & s_{j-1}
\end{vmatrix} =\\
&= (-1)^j
\begin{vmatrix}
a_0 & 0 & \dots & 0\\
a_1 & a_0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
a_{2j} & a_{2j-1} & \dots & a_0
\end{vmatrix}
\cdot
\begin{vmatrix}
1 & 0 & 0 & \dots & 0 & 0 & \dots & 0\\
0 & s_{-1} & s_0 & \dots & s_{j-2} & s_{j-1} & \dots & s_{2j-2}\\
0 & 1 & 0 & \dots & 0 & 0 & \dots & 0\\
0 & 0 & s_{-1} & \dots & s_{j-3} & s_{j-2} & \dots & s_{2j-3}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & s_{-1} & s_0 & \dots & s_{j-1}\\
0 & 0 & 0 & \dots & 1 & 0 & \dots & 0
\end{vmatrix} =\\
&= (-1)^j
\begin{vmatrix}
a_0 & a_1 & a_2 & \dots & a_{j-1} & a_j & \dots & a_{2j}\\
0 & b_0 & b_1 & \dots & b_{j-2} & b_{j-1} & \dots & b_{2j-1}\\
0 & a_0 & a_1 & \dots & a_{j-2} & a_{j-1} & \dots & a_{2j-1}\\
0 & 0 & b_0 & \dots & b_{j-3} & b_{j-2} & \dots & b_{2j-2}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & b_0 & b_1 & \dots & b_j\\
0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_j
\end{vmatrix} = a_0\nabla_{2j}(p,q).
\end{aligned}
$$

$\square$

Remark 1.6. Since the formulæ (1.13) are algebraic identities also valid for polynomials $\widetilde{p} = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$, and $\widetilde{q} = \sum_{j=0}^{n} b_j z^j$, they also hold when p and q are entire functions and R is a meromorphic function.

From (1.12) one can see that, whenever $b_0 \neq 0$, we have

$$\nabla_{2j}(p,q) = \nabla_{2j}(-q,p), \quad j = 1,2,\dots. \tag{1.15}$$

This observation, coupled with Theorem 1.5, yields the following well-known fact (cf. [27][1]).

Corollary 1.7. *For two infinite sequences $S := (s_j)_{j=-1}^{\infty}$, $T := (t_j)_{j=-1}^{\infty}$ with $s_{-1} \neq 0$ and $t_{-1} \neq 0$, the following conditions are equivalent:*

1)
$$
\begin{aligned}
s_{-1}t_{-1} + 1 &= 0,\\
s_i t_{-1} + s_{i-1}t_0 + \dots + s_0 t_{i-1} + s_{-1}t_i &= 0, \qquad i = 0,1,2,\dots.
\end{aligned}
$$

2)
$$D_k(S) = s_{-1}^{2k} D_k(T), \qquad k = 1,2,\dots,$$

where $D_k(S)$ and $D_k(T)$ are the Hankel minors (1.6) for the sequences S and T, respectively.

Proof. Let

$$R(z) = \frac{q(z)}{p(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots,$$

then

$$-\frac{1}{R(z)} = -\frac{p(z)}{q(z)} = t_{-1} + \frac{t_0}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots.$$

[1]It appears that this basic fact was known much earlier than the work of Edrei but we do not know of the original reference.

The assertion of the corollary follows from Theorem 1.5, formula (1.15) and the identity

$$\left(s_{-1}+\frac{s_0}{z}+\frac{s_1}{z^2}+\frac{s_2}{z^3}+\cdots\right)\left(t_{-1}+\frac{t_0}{z}+\frac{t_1}{z^2}+\frac{t_2}{z^3}+\cdots\right)\equiv -1.$$

$\square$

1.2 Resultant formulæ and applications: discriminant and Orlando formulæ

This section is devoted to the resultant, an important object that is closely related to the greatest common divisor of two polynomials. The discriminant of a single polynomial occurs as a special case of the resultant of that polynomial and its derivative. Moreover, the generalized Orlando formula connects the resultant of two polynomials, p and q, with the roots of the aggregate polynomial $h(z):=p(z^2)+zq(z^2)$. Although the resultant is classically known, this last connection is new.

Definition 1.8. Let two polynomials p and q be given in (1.2)–(1.3) with $n:=\deg p$ and $m:=\deg q$. The *resultant* of p and q is the following determinant of order $n+m$

$$\mathbf{R}(p,q):=\begin{vmatrix} a_0 & a_1 & \dots & a_{m-1} & a_m & \dots & a_{n-1} & a_n & \dots & a_{n+m-1}\\ 0 & a_0 & \dots & a_{m-2} & a_{m-1} & \dots & a_{n-2} & a_{n-1} & \dots & a_{n+m-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_0 & a_1 & \dots & a_{n-m} & a_{n-m+1} & \dots & a_n\\ b_{n-m} & b_{n-m+1} & \dots & b_{n-1} & b_n & \dots & b_{2n-m-1} & b_{2n-m} & \dots & b_{2n-1}\\ 0 & b_{n-m} & \dots & b_{n-2} & b_{n-1} & \dots & b_{2n-m-2} & b_{2n-m-1} & \dots & b_{2n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & 0 & \dots & b_{n-m} & b_{n-m+1} & \dots & b_n \end{vmatrix}, \tag{1.16}$$

where we set $a_i:=0$ and $b_i:=0$ for all $i>n$.

From (1.16) it can be immediately seen that

$$\mathbf{R}(p,q)=(-1)^{nm}\mathbf{R}(q,p).$$

The resultant of two polynomials is a multi-affine function of the roots of these polynomials, as the following formula shows.

Theorem 1.9. *Given polynomials p and q as in (1.2)–(1.3) with $b_0\neq 0$, let λ_i $(i=1,\dots,n)$ denote the zeros of p, and let μ_j $(j=1,\dots,n)$ denote the zeros of q. Then*

$$\mathbf{R}(p,q)=(-1)^{\frac{n(n-1)}{2}}\nabla_{2n}(p,q)=a_0^n\prod_{i=1}^{n}q(\lambda_i)=a_0^nb_0^n\prod_{i,j=1}^{n}(\lambda_i-\mu_j)=(-1)^nb_0^n\prod_{j=1}^{n}p(\mu_j). \tag{1.17}$$

Proof. At first, assume that all roots of the polynomial p are distinct. Then the function R admits the representation

$$R(z)=\beta+\sum_{j=1}^{n}\frac{\alpha_j}{z-\lambda_j}, \tag{1.18}$$

where

$$\alpha_j=\frac{p(\lambda_j)}{q'(\lambda_j)},\quad j=1,\dots,n. \tag{1.19}$$

Here $\lambda_j\neq\lambda_i$ whenever $i\neq j$, according to the assumption. Comparing the representation (1.18) with the expansion (1.4), we obtain

$$s_k=\sum_{i=1}^{n}\alpha_i\lambda_i^k,\qquad k=0,1,2,\dots \tag{1.20}$$

and $s_{-1} = \dfrac{b_0}{a_0} \neq 0$. From (1.12), (1.13) and (1.16) it follows that

$$\mathbf{R}(p,q) = (-1)^{\frac{n(n-1)}{2}} \nabla_{2n}(p,q) = (-1)^{\frac{n(n-1)}{2}} a_0^{2n} D_n(R). \tag{1.21}$$

The formulæ (1.20) yield

$$\begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \\ \alpha_1\lambda_1 & \alpha_2\lambda_2 & \alpha_3\lambda_3 & \dots & \alpha_n\lambda_n \\ \alpha_1\lambda_1^2 & \alpha_2\lambda_2^2 & \alpha_3\lambda_3^2 & \dots & \alpha_n\lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1\lambda_1^{n-1} & \alpha_2\lambda_2^{n-1} & \alpha_3\lambda_3^{n-1} & \dots & \alpha_n\lambda_n^{n-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{pmatrix}.$$

This equality implies

$$D_n(R) = \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} \end{vmatrix} = \prod_{i=1}^{n} \alpha_i \cdot \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}^2 \tag{1.22}$$

This formula combined with the residue formula (1.19) implies

$$D_n(R) = \prod_{i=1}^{n} \frac{q(\lambda_i)}{p'(\lambda_i)} \cdot \prod_{j<i} (\lambda_i - \lambda_j)^2. \tag{1.23}$$

Since

$$p'(z) = a_0 \sum_{i=1}^{n} \prod_{\substack{k=1 \\ k\neq i}}^{n} (z - \lambda_k),$$

we have

$$\prod_{i=1}^{n} p'(\lambda_i) = a_0^n \prod_{i=1}^{n} \prod_{\substack{k=1 \\ k\neq i}}^{n} (\lambda_i - \lambda_k) = a_0^n \prod_{j<i} (\lambda_i - \lambda_j) \prod_{i<j} (\lambda_i - \lambda_j).$$

This product has exactly $n(n-1)$ factors of the form $\lambda_i - \lambda_j$, therefore

$$\prod_{i=1}^{n} p'(\lambda_i) = a_0^n (-1)^{\frac{n(n-1)}{2}} \prod_{j<i} (\lambda_i - \lambda_j)^2. \tag{1.24}$$

Now from (1.23)–(1.24) we obtain

$$D_n(R) = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_0^n} \prod_{i=1}^{n} q(\lambda_i) = (-1)^{\frac{n(n-1)}{2}} \cdot \frac{b_0^n}{a_0^n} \prod_{i,j=1}^{n} (\lambda_i - \mu_j). \tag{1.25}$$

The formula (1.17) follows from (1.21) and (1.25).

If the polynomial p has multiple zeros, we can consider an approximating polynomial p_ε with simple zeros such that

$$\lim_{\varepsilon\to 0} p_\varepsilon(z) = p(z) \quad \text{for all } \ z.$$

Then

$$\nabla_{2n}(R_\varepsilon) \xrightarrow[\varepsilon\to 0]{} \nabla_{2n}(R),$$

where $R_\varepsilon(z) := \dfrac{q(z)}{p_\varepsilon(z)}$. The formula (1.17) is valid for the polynomials q and p_ε, so it is also valid at the limit, i.e., for the polynomials q and p. ∎

Corollary 1.10. *Under the conditions of Theorem 1.9, let* $\deg q = m \le n$. *Then*

$$\begin{array}{rcccccl} \mathbf{R}(p,q) & = & (-1)^{\frac{n(n-1)}{2}} a_0^{m-n}\nabla_{2n}(p,q) & = & a_0^m \displaystyle\prod_{i=1}^{n} q(\lambda_i) & = & \\ = \; a_0^m b_{n-m}^n \displaystyle\prod_{i=1}^{n}\prod_{j=1}^{m}(\lambda_i-\mu_j) & = & (-1)^{nm} b_{n-m}^n \displaystyle\prod_{j=1}^{m} p(\mu_j) & = & (-1)^{nm} R(q,p). & & \end{array} \tag{1.26}$$

Proof. If $\deg q = m (\le n)$, then the first nonzero coefficient of q is b_{n-m}, according to (1.3). Therefore,

$$\nabla_{2n}(p,q) = (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} a_0 & a_1 & \dots & a_{n-m-1} & a_{n-m} & \dots & a_n & \dots & a_{2n-m-2} & \dots & a_{2n-1} \\ 0 & a_0 & \dots & a_{n-m-2} & a_{n-m-1} & \dots & a_{n-1} & \dots & a_{2n-m-3} & \dots & a_{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_0 & \dots & a_m & \dots & a_{n-2} & \dots & a_{n+m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & a_0 & \dots & a_{n-m} & \dots & a_n \\ 0 & 0 & \dots & 0 & b_{n-m} & \dots & b_n & \dots & b_{2n-m-1} & \dots & b_{2n-1} \\ 0 & 0 & \dots & 0 & 0 & \dots & b_{n-1} & \dots & b_{2n-m-2} & \dots & b_{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & b_{n-m} & \dots & b_n \end{vmatrix}.$$

Thus, we have

$$\nabla_{2n}(p,q) = (-1)^{\frac{n(n-1)}{2}} a_0^{n-m}\mathbf{R}(p,q).$$

This relation and the formulæ (1.13), (1.25) yield (1.26) when p has only simple roots. But (1.26) is also valid when p has multiple zeros, which can be proved by an approximation argument as above. ∎

The formulæ (1.17) and (1.26) now imply the well-known property of the resultant:

Corollary 1.11. $\mathbf{R}(p,q) = 0$ *if and only if the polynomials* p *and* q *have common roots.*

Next, we consider a function that allows us to test whether a single polynomial has multiple roots.

Definition 1.12. Given a polynomial (1.2) with roots λ_i $(i = 1, \dots, n)$, its *discriminant* is defined as

$$\mathbf{D}(p) := a_0^{2n-2}\prod_{j<i}^{n}(\lambda_i-\lambda_j)^2. \tag{1.27}$$

It is clear from (1.27) that the discriminant of a polynomial is equal to zero if and only if the polynomial has multiple zeros. But multiple zeros of a polynomial are the zeros that it shares with its derivative. The following connection between the discriminant of p and the resultant of p and p' should not come as a surprise.

Theorem 1.13. *Given a polynomial* (1.2) *of degree* n, *we have*

$$\mathbf{R}(p,p') = (-1)^{\frac{n(n-1)}{2}} a_0\mathbf{D}(p). \tag{1.28}$$

Proof. Indeed, the resultant $\mathbf{R}(p, p')$ satisfies (1.26). Together with (1.24) and (1.27), it gives

$$\mathbf{R}(p,p') = a_0^{n-1}\prod_{i=1}^{n} p'(\lambda_i) = a_0^{2n-1}(-1)^{\frac{n(n-1)}{2}}\prod_{j<i}(\lambda_i-\lambda_j)^2 = (-1)^{\frac{n(n-1)}{2}} a_0\mathbf{D}(p).$$

□

From (1.26) we obtain

$$\nabla_{2n}(p,p') = (-1)^{\frac{n(n-1)}{2}} a_0\mathbf{R}(p,p') = a_0^2\mathbf{D}(p).$$

Thus, the discriminant of the polynomial p is the following $(2n-1)\times(2n-1)$ determinant:

$$\mathbf{D}(p) = \frac{1}{a_0}\begin{vmatrix} na_0 & (n-1)a_1 & (n-2)a_2 & \dots & a_{n-1} & 0 & \dots & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & \dots & 0 \\ 0 & na_0 & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & \dots & 0 \\ 0 & 0 & na_0 & \dots & 3a_{n-3} & 2a_{n-2} & \dots & 0 \\ 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 & \dots & a_n \\ 0 & 0 & 0 & \dots & na_0 & (n-1)a_1 & \dots & a_{n-1} \end{vmatrix} \tag{1.29}$$

We now give one more formula for the resultant, which is very close to the well-known Orlando formula [73] (see also [36]). More precisely, the application of this formula to the resultant of the odd and the even parts of a polynomial yields exactly the Orlando formula.

Theorem 1.14. *Let the polynomials p and q be given by (1.2)–(1.3), with $\deg p = n$ and $\deg q = m \leq n-1$. Then the resultant of these polynomials can be computed as follows:*

$$\mathbf{R}(p,q) = (-1)^{\frac{n(n-1)}{2}} a_0^{m+n}\prod_{1\leq i<k\leq 2n}(z_i+z_k), \tag{1.30}$$

where z_i $(i = 1,\dots,2n)$ are the zeros of the following polynomial of degree $2n$

$$h(z) = p(z^2) + zq(z^2). \tag{1.31}$$

Proof. Let λ_i $(i = 1,\dots,n)$ be the zeros of the polynomial p, and let μ_j $(j = 1,\dots,m)$ be the zeros of the polynomial q. From (1.31) it follows that

$$p(z^2) = \frac{h(z)+h(-z)}{2}, \qquad q(z^2) = \frac{h(z)-h(-z)}{2z}. \tag{1.32}$$

and

$$p(\mu_j) = h(\pm\sqrt{\mu_j}), \qquad j = 1,\dots,m, \qquad q(\lambda_i) = \pm\frac{h(\pm\sqrt{\lambda_i})}{\sqrt{\lambda_i}}, \qquad i = 1,\dots,n. \tag{1.33}$$

From (1.32) we obtain

$$p(z_k^2) = \frac{h(-z_k)}{2}, \qquad q(z_k^2) = -\frac{h(-z_k)}{2z_k}, \qquad k = 1,\dots,2n, \tag{1.34}$$

where z_k are the zeros of the polynomial h. Since $\deg q = m$ by assumption, we conclude that $b_{n-m} \neq 0$ but $b_0 = \ldots = b_{n-m-1} = 0$. Thus, (1.26) implies

$$R(p,q) = b_{n-m}^n\prod_{j=1}^{m} p(\mu_j). \tag{1.35}$$

Using (1.35), (1.33) and (1.34), we have

$$\begin{aligned}(\mathbf{R}(p,q))^2 &= b_{n-m}^{2n}\prod_{j=1}^{m} h(\sqrt{\mu_j})h(-\sqrt{\mu_j}) = b_{n-m}^{2n}\prod_{j=1}^{m} a_0^2\prod_{k=1}^{2n}(\mu_j - z_k^2) = a_0^{2m}\prod_{k=1}^{2n}\left[b_{n-m}\prod_{j=1}^{m}(z_k^2-\mu_j)\right]\\ &= a_0^{2m}\prod_{k=1}^{2n} q(z_k^2) = a_0^{2m}\prod_{k=1}^{2n}\frac{h(-z_k)}{2z_k} = a_0^{2m}\prod_{k=1}^{2n}\frac{a_0}{2z_k}\prod_{i=1}^{2n}(z_i+z_k)\\ &= a_0^{2m+2n}\prod_{k=1}^{2n}\prod_{\substack{i=1\\ i\neq k}}^{2n}(z_i+z_k) = a_0^{2m+2n}\prod_{1\le i<k\le 2n}(z_i+z_k)^2.\end{aligned}$$

Thus, we obtain

$$\mathbf{R}(p,q) = \pm a_0^{m+n}\prod_{1\le i<k\le 2n}(z_i+z_k).$$

To determine the sign, we consider the special case $h(z) = (z-1)^{2n}$. Then

$$p(z^2) = \frac{(z-1)^{2n}+(z+1)^{2n}}{2}, \qquad q(z^2) = \frac{(z-1)^{2n}-(z+1)^{2n}}{2z},$$

so that $\deg p = n$, $\deg q = n-1$. The roots of the odd part q can be determined from the equation $(z-1)^{2n} = (z+1)^{2n}$ except that the zero root should be discarded. This shows that the roots of q are $\{w_k^2 : k = 1,\dots,n-1\}$ where w_k is defined from the equation $2/(w_k+1) = 1 - e^{\frac{\pi i k}{n}}$, $k = 1,\dots,n-1$. Consequently,

$$\begin{aligned}\prod_{k=1}^{n-1} p(w_k^2) &= \prod_{k=1}^{n-1}\left(\frac{2}{1-e^{\frac{\pi i k}{n}}}\right)^{2n} = \prod_{k=1}^{n-1}\left(\frac{ie^{\frac{-\pi i k}{2n}}}{\sin\left(\frac{\pi k}{n}\right)}\right)^{2n}\\ &= i^{2n(n-1)}\prod_{k=1}^{n-1} e^{-\pi i k}\frac{1}{\sin^{2n}\left(\frac{\pi k}{n}\right)} = (-1)^{\sum_{i=1}^{n-1} k}\prod_{k=1}^{n-1}\frac{1}{\sin^{2n}\left(\frac{\pi k}{n}\right)}.\end{aligned}$$

Thus, according to the last formula in (1.26), the sign of the resultant $R(p,q)$ in our special case is

$$(-1)^{n(n-1)}\operatorname{sign} b_1^n(-1)^{\frac{n(n-1)}{2}} = (-1)^{\frac{n(n+1)}{2}} = (-1)^{n+\frac{n(n-1)}{2}}.$$

As we already established, the resultant of p and q is a polynomial in the coefficients of p and q, hence in the coefficients of h. The expression $(-1)^{\frac{n(n-1)}{2}}a_0^{2n-1}\prod_{i<k}(z_i+z_k)$ is a symmetric function of the roots of h multiplied by its leading coefficient to the power $2n-1$, and hence also a polynomial in the coefficients of h. Since the two polynomials must be identically equal, we conclude that the sign $(-1)^{\frac{n(n-1)}{2}}$ occurs at all times whenever $\deg p = n$, $\deg q = n-1$.

We now show how to produce a formula for the case $m < n-1$ from the already established formula for $m = n-1$. Given a polynomial q of degree $n-1$, set b_1 through b_{n-m-1} to zero, thus obtaining a polynomial of degree m. Observe what happens to the determinantal expression (1.16). Since the lower left block of size $n\times(n-m-1)$ is now filled with zeros, the upper-triangular submatrix above produces the factorization

$$\widetilde{\mathbf{R}}(p,q) = a_0^{n-m-1}\mathbf{R}(p,q),$$

where $\widetilde{\mathbf{R}}(p,q)$ denotes the “old” resultant of p and q constructed as if $\deg q$ were equal to $(n-1)$. Thus the “new” resultant $\mathbf{R}(p,q)$ satisfies the equation

$$a_0^{n-m-1}\mathbf{R}(p,q) = (-1)^{\frac{n(n-1)}{2}}a_0^{2n-1}\prod_{1\le i<k\le 2n}(z_i+z_j),$$

hence $\mathbf{R}(p,q) = (-1)^{\frac{n(n-1)}{2}}a_0^{n+m}\prod_{1\le i<k\le 2n}(z_i+z_j)$. □

Our next statement can be proved analogously.

Theorem 1.15. *Let the polynomials p and q be given by (1.2)–(1.3), and let $\deg q = m \leq n = \deg p$. Then the resultant of these polynomials can be computed by the formula*

$$\mathbf{R}(p,q) = (-1)^{\frac{n(n-1)}{2}} a_0^{m+n} \prod_{1\leq i<k\leq 2n+1} (z_i + z_k), \tag{1.36}$$

where z_i $(i = 1, \ldots, 2n+1)$ are the zeros of the polynomial

$$g(z) = q(z^2) + zp(z^2).$$

The famous Orlando formula is a simple consequence of Theorems 1.14–1.15. Before proving the Orlando formula, we introduce the following determinants for the polynomial (1.2)

$$\Delta_j(p) = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \dots & a_{2j-1} \\ a_0 & a_2 & a_4 & a_6 & \dots & a_{2j-2} \\ 0 & a_1 & a_3 & a_5 & \dots & a_{2j-3} \\ 0 & a_0 & a_2 & a_4 & \dots & a_{2j-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_j \end{vmatrix}, \quad j = 1, \ldots, n, \tag{1.37}$$

where we set $a_i := 0$ for $i > n$.

Definition 1.16. The determinants $\Delta_j(p)$ $(j = 1, \ldots, n)$ are called *the Hurwitz determinants* or *the Hurwitz minors* of the polynomial p.

It is easy to see that the polynomial p can be always represented as follows

$$p(z) = p_0(z^2) + zp_1(z^2), \tag{1.38}$$

where

$$\begin{aligned} p_0(u) &= a_1u^l + a_3u^{l-1} + \cdots + a_n, \\ p_1(u) &= a_0u^l + a_2u^{l-1} + \cdots + a_{n-1}, \end{aligned} \tag{1.39}$$

if the degree n of the polynomial $p(z)$ is odd: $n = 2l + 1$, and

$$\begin{aligned} p_0(u) &= a_0u^l + a_2u^{l-1} + \cdots + a_n, \\ p_1(u) &= a_1u^{l-1} + a_3u^{l-2} + \cdots + a_{n-1}, \end{aligned} \tag{1.40}$$

if $n = 2l$.

Theorem 1.17 (Orlando, [73, 36])**.** *Let the polynomial p of degree n be given by (1.2). Then the determinant $\Delta_{n-1}(p)$ defined by (1.37) can be computed from the formula*

$$\Delta_{n-1}(p) = (-1)^{\frac{n(n-1)}{2}} a_0^{n-1} \prod_{1\leq i<j\leq n} (z_i + z_j), \tag{1.41}$$

where z_i, $i = 1, \ldots, n$, are the zeros of the polynomial p.

This equality is known as *the Orlando formula*.

Proof. At first, let the degree n of p be odd, $n = 2l + 1$. Then (1.38)–(1.39) show that $\deg p_0 \leq l$, $\deg p_1 = l$, and the leading coefficient of p_1 is a_0. Thus, (1.12), (1.26) and (1.36) imply

$$\Delta_{n-1}(p) = (-1)^l \nabla_{2l}(p_1, p_0) = (-1)^{\frac{l(l+1)}{2}} a_0^{l-\deg p_0} \mathbf{R}(p_1, p_0) = (-1)^l a_0^{2l} \prod_{1\leq i<j\leq 2l+1} (z_i + z_j),$$

which coincides with (1.41) since $n = 2l + 1$.

If $n = 2l$, then (1.38) and (1.40) imply $\deg p_1 \leq l-1$, $\deg p_0 = l$, and the leading coefficient of the polynomial p_0 is a_0. As above, the formulæ (1.12), (1.26) and (1.30) can now be combined to obtain

$$\Delta_{n-1}(p) = a_0^{-1}\nabla_{2l}(p_0,p_1) = (-1)^{\frac{l(l-1)}{2}} a_0^{l-\deg p_1-1}\mathbf{R}(p_0,p_1) = (-1)^l a_0^{2l-1}\prod_{1\leq i<j\leq 2l}(z_i+z_j),$$

which is exactly the formula (1.41). ◻

1.3 Euclidean algorithm, the greatest common divisor, and continued fractions: general case

In the previous subsection, we considered the resultant of two polynomials and observed that it is equal to zero if and only if these polynomials have a nontrivial common divisor. The standard way to find their greatest common divisor is via the Euclidean algorithm.

Let us consider polynomials p and q given by (1.2)–(1.3) and let us denote[2]

$$f_0(z) := p(z), \qquad f_1(z) := q(z) - \frac{b_0}{a_0}p(z).$$

Construct a sequence of polynomials $f_0, f_1, \ldots, f_k$ $(k \leq n)$ by the following formula

$$f_{j-1}(z) = q_j(z)f_j(z) + f_{j+1}(z), \quad j = 1,\ldots,k \quad (f_{k+1}(z) = 0), \tag{1.42}$$

where q_j is the quotient and f_{j+1} is the remainder from the division of f_{j-1} by f_j. The last polynomial in this sequence, f_k, is the greatest common divisor of the polynomials f_0 and f_1 (and also all other polynomials f_j in the sequence). In other words,

$$f_j(z) = h_j(z)f_k(z), \quad j = 0,1,\ldots,k, \tag{1.43}$$

where $h_k(z) = 1$. Now, denote

$$R_j(z) := \frac{f_j(z)}{f_{j-1}(z)} = \frac{h_j(z)}{h_{j-1}(z)}. \tag{1.44}$$

Rewriting (1.42), we obtain

$$R_j(z) = \frac{1}{q_j(z) + R_{j+1}(z)}, \quad j = 1,\ldots,k, \tag{1.45}$$

where $R_{k+1}(z) \equiv 0$. Using this equality, we can represent the function $R_1(z)$ as a continued fraction[3]:

$$R_1(z) = \frac{f_1(z)}{f_0(z)} = \frac{h_1(z)}{h_0(z)} = \cfrac{1}{q_1(z) + \cfrac{1}{q_2(z) + \cfrac{1}{q_3(z) + \cfrac{1}{\ddots + \cfrac{1}{q_k(z)}}}}}. \tag{1.46}$$

It is easy to see that, for each $j = 0,\ldots,k-1$, the polynomial h_j is the leading principal minor of order $k-j$ of the following $k\times k$ tridiagonal matrix

$$\mathcal{J}(z) = \begin{pmatrix} q_k(z) & -1 & 0 & \ldots & 0 & 0 \\ 1 & q_{k-1}(z) & -1 & \ldots & 0 & 0 \\ 0 & 1 & q_{k-2}(z) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_2(z) & -1 \\ 0 & 0 & 0 & \cdots & 1 & q_1(z) \end{pmatrix}. \tag{1.47}$$

[2]Thus, $\deg f_1 < \deg f_0$. Obviously, if $\deg q < \deg p$, that is, if $b_0 = 0$, then $f_1 = q$.
[3]The functions $R = q/p$ and R_1 are related via the formula $R(z) = R_1(z) + b_0/a_0$.

In particular, $h_0(z) = \det \mathcal{J}(z)$. Thus, the determinant of the matrix $\mathcal{J}(z)$ is the denominator of $R(z)$. This is a very useful observation, as certain properties of the function R turn out to be connected to the location of the eigenvalues of the generalized (matrix polynomial) eigenvalue problem

$$\mathcal{J}(z)u = 0. \tag{1.48}$$

Conversely, the behavior of the eigenvalues of the problem (1.48) can provide information about certain properties of the function R. Later in this paper we will give examples of such interrelation.

More generally, in addition to the fraction (1.46) we may consider the fractions

$$R_j(z) = \frac{h_j(z)}{h_{j-1}(z)} = \cfrac{1}{q_j(z) + \cfrac{1}{q_{j+1}(z) + \cfrac{1}{q_{j+2}(z) + \cfrac{1}{\ddots + \cfrac{1}{q_k(z)}}}}}, \quad j = 1, \dots, k.$$

The continued fraction expansion (1.46) can be found efficiently in terms of the corresponding Hankel minors, as the next theorem shows.

Theorem 1.18. *Two rational functions R and G vanishing at ∞ are related via the identity*

$$R(z) = \frac{1}{g(z) + G(z)}, \tag{1.49}$$

where g is a polynomial of degree $m(\geq 1)$ with leading coefficient α

$$g(z) = \alpha z^m + \cdots \tag{1.50}$$

if and only if

$$D_1(R) = D_2(R) = \cdots = D_{m-1}(R) = 0, \quad (\textit{when} \quad m > 1) \tag{1.51}$$

and

$$D_{m+j}(R) = (-1)^{\frac{m(m-1)}{2}} \cdot \frac{(-1)^j D_j(G)}{\alpha^{m+2j}}, \quad j = 0, 1, 2, \dots, \tag{1.52}$$

where $D_0(G) := 1$, and the determinants $D_j(R)$ and $D_j(G)$ are defined by (1.6).

Proof. Since the functions R and G vanish at ∞, they can be expanded into Laurent series

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots + \frac{s_{m-1}}{z^m} + \frac{s_m}{z^{m+1}} + \cdots, \qquad G(z) = \frac{t_0}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots. \tag{1.53}$$

Moreover, the conditions (1.49)–(1.50) hold if and only if

$$s_0 = s_1 = \cdots = s_{m-2} = 0 \quad \text{and} \quad s_{m-1} \neq 0. \tag{1.54}$$

In fact, if R satisfies (1.49)–(1.50), then

$$s_i = \lim_{z\to\infty} z^{i+1} R(z) = \lim_{z\to\infty} \frac{1}{\dfrac{g(z)}{z^{i+1}} + \dfrac{G(z)}{z^{i+1}}} = 0, \quad i = 0, 1, 2, \dots, m-2. \tag{1.55}$$

and

$$s_{m-1} = \lim_{z\to\infty} z^m R(z) = \lim_{z\to\infty} \frac{1}{\alpha + \dfrac{\gamma_1 z^{m-1} + \cdots + \gamma_m}{z^m} + \dfrac{G(z)}{z^m}} = \frac{1}{\alpha} \neq 0. \tag{1.56}$$

Now assume that the condition (1.54) holds and that $R(z) = \dfrac{1}{f(z) + G(z)}$ for some polynomial f. If $\deg f = j$ for some $1 \leq j < m$, that is[4], if $f(z) = \gamma z^j + \cdots$, where $\gamma \neq 0$, then $s_{j-1} = \lim\limits_{z\to\infty} z^j R(z) = \dfrac{1}{\gamma} \neq 0$,

[4]The degree j of f cannot be 0, since the limit $\lim\limits_{z\to\infty} R(z) = \dfrac{1}{\gamma}$ is nonzero for a constant nonzero function $f(z) = \gamma$.

contrary to (1.54). On the other hand, if the degree of f is greater than m, then $s_{m-1} = \lim_{z\to\infty} z^m R(z) = 0$. Thus, f must be of exact degree m since otherwise a contradiction arises with one of the conditions (1.54).

Also note that the equalities (1.54) are equivalent to (1.51). Moreover, from (1.54) we have

$$D_m(R) = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & s_{m-1} \\ 0 & 0 & 0 & \dots & s_{m-1} & s_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & s_{m-1} & s_m & \dots & s_{2m-4} & s_{2m-3} \\ s_{m-1} & s_m & s_{m+1} & \dots & s_{2m-3} & s_{2m-2} \end{vmatrix} = (-1)^{\frac{m(m-1)}{2}} s_{m-1}^m \neq 0. \tag{1.57}$$

For convenience, denote the coefficients of the polynomial g as follows

$$g(z) =: \alpha z^m + t_{-m} z^{m-1} + t_{-m+1} z^{m-2} + \dots + t_{-2} z + t_{-1}.$$

If the functions R and G satisfy (1.49)–(1.50), then, according to (1.53)–(1.54), we have

$$\left[\frac{s_{m-1}}{z^m} + \frac{s_m}{z^{m+1}} + \frac{s_{m+1}}{z^{m+2}} + \cdots\right]\left[\alpha z^m + t_{-m} z^{m-1} + \dots + t_{-1} + \frac{t_0}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots\right] \equiv 1.$$

This identity implies the following relations:

$$\begin{aligned} s_{m-1} &= \frac{1}{\alpha}, \\ s_{m+j} &= -\sum_{i=0}^{j} \frac{t_{-m+i}}{\alpha} \cdot s_{m+j-i-1}, \qquad j = 0, 1, 2, \dots \end{aligned} \tag{1.58}$$

Now, the equality (1.52) for $D_m(R)$ follows from (1.57)–(1.58).

For a fixed number $j \geq 1$, consider the determinant $D_{m+j}(R)$:

$$D_{m+j}(R) = \begin{vmatrix} 0 & 0 & \dots & 0 & s_{m-1} & s_m & \dots & s_{m+j-1} \\ 0 & 0 & \dots & s_{m-1} & s_m & s_{m+1} & \dots & s_{m+j} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & s_{m-1} & \dots & s_{2m-4} & s_{2m-3} & s_{2m-2} & \dots & s_{2m+j-3} \\ s_{m-1} & s_m & \dots & s_{2m-3} & s_{2m-2} & s_{2m-1} & \dots & s_{2m+j-2} \\ s_m & s_{m+1} & \dots & s_{2m-2} & s_{2m-1} & s_{2m} & \dots & s_{2m+j-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m+j-1} & s_{m+j} & \dots & s_{2m+j-3} & s_{2m+j-2} & s_{2m+j-1} & \dots & s_{2m+2j-2} \end{vmatrix}. \tag{1.59}$$

Add to each i^{th} column ($i = m{+}j, m{+}j{-}1, \dots, 3, 2$) columns $i{-}1, i{-}2, \dots, 1$ multiplied by $\frac{t_{-m}}{\alpha}, \frac{t_{-m+1}}{\alpha}, \dots, \frac{t_{j-3}}{\alpha}, \frac{t_{j-2}}{\alpha}$, respectively. This eliminates the entries in the upper right corner of the determinant (1.59) using (1.58). So, we can rewrite the original determinant as a product of the following two determinants of order m and j, respectively:

$$D_{m+j}(R) = \begin{vmatrix} 0 & 0 & \dots & 0 & s_{m-1} \\ 0 & 0 & \dots & s_{m-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & s_{m-1} & \dots & 0 & 0 \\ s_{m-1} & 0 & \dots & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} d_{11} & d_{12} & \dots & d_{1,j-1} & d_{1,j} \\ d_{21} & d_{22} & \dots & d_{2,j-1} & d_{2,j} \\ d_{31} & d_{32} & \dots & d_{3,j-1} & d_{3,j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{j1} & d_{j2} & \dots & d_{j,j-1} & d_{jj} \end{vmatrix},$$

where

$$d_{i_1,i_2} := -s_{m+i_1-2} \cdot \frac{t_{i_2-1}}{\alpha} - s_{m+i_1-3} \cdot \frac{t_{i_2}}{\alpha} - \dots - s_m \cdot \frac{t_{i_1+i_2-3}}{\alpha} - s_{m-1} \cdot \frac{t_{i_1+i_2-2}}{\alpha}. \tag{1.60}$$

From (1.58) and (1.60) we obtain

$$D_{m+j}(R) = (-1)^{\frac{m(m-1)}{2}} \cdot \frac{(-1)^j}{\alpha^{m+j}} \begin{vmatrix} s_{m-1} & 0 & \dots & 0 & 0 \\ s_m & s_{m-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{m+j-3} & s_{m+j-4} & \dots & s_{m-1} & 0 \\ s_{m+j-2} & s_{m+j-3} & \dots & s_m & s_{m-1} \end{vmatrix} \cdot \begin{vmatrix} t_0 & t_1 & \dots & t_{j-2} & t_{j-1} \\ t_1 & t_2 & \dots & t_{j-1} & t_j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{j-2} & t_{j-1} & \dots & t_{2j-4} & t_{2j-3} \\ t_{j-1} & t_j & \dots & t_{2j-3} & t_{2j-2} \end{vmatrix}.$$

Conversely, if the equalities (1.51)–(1.52) hold, then we obtain $s_0 = s_1 = \cdots = s_{m-2} = 0$ from (1.51) by induction, which was already proved to be equivalent to the fact that R satisfies (1.49), with a polynomial g of degree at least m and $D_m(R) = (-1)^{\frac{m(m-1)}{2}} s_{m-1}^m$. If $\deg g > m$, then

$$s_{m-1} = \lim_{z\to\infty} z^m R(z) = \lim_{z\to\infty} \frac{1}{\dfrac{g(z)}{z^m} + \dfrac{G(z)}{z^m}} = 0,$$

therefore, $D_m(R) = (-1)^{\frac{m(m-1)}{2}} s_{m-1}^m = (-1)^{\frac{m(m-1)}{2}} \dfrac{1}{\alpha^m} = 0$ and $\dfrac{1}{\alpha} = 0$, according to (1.52). Thus, from (1.51)–(1.52) we get $D_i(R) = 0$ for all $i \in \mathbb{N}$, which is impossible, since R is a rational function, hence at least one of the minors $D_i(R)$ must be nonzero due to Theorem 1.2. Consequently, the polynomial g satisfies (1.50) with $\alpha \neq 0$. ∎

If a rational function R with exactly r poles is expanded into a continued fraction (1.46), then the rational functions R_j defined in (1.44) satisfy the relations (1.45), where

$$q_j(z) = \alpha_j z^{n_j} + \cdots, \qquad \alpha_j \neq 0, \quad j = 1, 2, \ldots, k. \tag{1.61}$$

Here $n_1 + n_2 + \cdots + n_k = r$ and $n_1 \geq 1$, since $\deg f_1 < \deg f_0$, as is remarked above. The other degrees n_i are greater or equal to 1 due to the structure of the Euclidean algorithm (1.42).

From (1.51)–(1.52) we obtain, for a fixed integer j ($j = 1, 2, \ldots, k$), the following product formulæ:

$$\begin{aligned} D_{n_1+n_2+\cdots+n_j}(R_1) &= (-1)^{\frac{n_1(n_1-1)}{2}} (-1)^{\sum\limits_{i=2}^{j} n_i} \cdot \frac{1}{\alpha_1^{n_1+2\sum\limits_{i=2}^{j} n_i}} \cdot D_{n_2+n_3+\cdots+n_j}(R_2) = \\ &= (-1)^{\frac{n_1(n_1-1)}{2}} (-1)^{\sum\limits_{i=2}^{j} n_i} \cdot \frac{1}{\alpha_1^{n_1+2\sum\limits_{i=2}^{j} n_i}} \cdot (-1)^{\frac{n_2(n_2-1)}{2}} (-1)^{\sum\limits_{i=3}^{j} n_i} \times \\ &\quad \times \frac{1}{\alpha_2^{n_2+2\sum\limits_{i=3}^{j} n_i}} \cdot D_{n_3+n_4+\cdots+n_j}(R_3) = \cdots \end{aligned}$$

This chain of equalities results in the formula

$$D_{n_1+n_2+\cdots+n_j}(R) = \prod_{i=1}^{j} (-1)^{\frac{n_i(n_i-1)}{2}} \cdot (-1)^{\sum\limits_{i=0}^{j-1} i n_{i+1}} \cdot \prod_{i=1}^{j} \frac{1}{\alpha_i^{n_i+2\sum\limits_{\rho=i+1}^{j} n_\rho}}, \quad j = 1, 2, \ldots, k. \tag{1.62}$$

Remark 1.19. Our discussion above can be summarized as follows: Suppose that a rational function R with r poles has a continued fraction expansion (1.46) with polynomials q_j satisfying (1.61). Then Theorems 1.2, 1.3 and 1.18 and formulæ (1.62) imply that all Hankel minors $D_j(R)$ are equal to zero, except for the minors[5] $D_{n_1}(R), D_{n_1+n_2}(R), \ldots, D_{n_1+n_2+\cdots+n_k}(R)$, which are not zero and which can be calculated from the formulæ (1.62).

Using Theorem 1.18 and the formulæ (1.62), we now describe equivalence classes of rational functions whose sequences of Hankel minors are the same.

Theorem 1.20. *Two rational functions*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots \quad \textit{and} \quad G(z) = \frac{t_0}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots\ ,$$

[5] Recall that $n_1 + n_2 + \cdots + n_k = r$.

both vanishing at infinity, have equal Hankel minors

$$D_j(R) = D_j(G), \qquad j = 1, 2, \dots\ , \tag{1.63}$$

if and only if their continued fraction expansions

$$R(z) = \cfrac{1}{q_1(z) + \cfrac{1}{q_2(z) + \cfrac{1}{q_3(z) + \cfrac{1}{\ddots + \cfrac{1}{q_{k_1}(z)}}}}} \quad \textit{and} \quad G(z) = \cfrac{1}{\widetilde{q}_1(z) + \cfrac{1}{\widetilde{q}_2(z) + \cfrac{1}{\widetilde{q}_3(z) + \cfrac{1}{\ddots + \cfrac{1}{\widetilde{q}_{k_2}(z)}}}}}$$

satisfy

$$k_1 = k_2 =: k, \tag{1.64}$$

and the polynomials q_j *and* $\widetilde{q}_j$*, for each* j $(j = 1, 2, \dots, k)$*, have equal degrees and equal leading coefficients:*

$$\begin{array}{l} q_j(z) = \alpha_j z^{n_j} + \cdots\ , \\ \widetilde{q}_j(z) = \alpha_j z^{n_j} + \cdots\ , \end{array} \qquad j = 1, 2, \dots, k. \tag{1.65}$$

Proof. If the continued fraction expansions of the functions R and G satisfy the conditions (1.64)–(1.65), then the equalities (1.63) follow from Remark 1.19 and the formulæ (1.62).

Conversely, let the Hankel minors associated with the functions R and G satisfy (1.63). Therefore, by Theorems 1.2 and 1.3, the functions R and G have equal number of poles. Moreover, they can be then represented as follows:

$$R(z) = R_1(z) = \frac{1}{q_1(z) + R_2(z)} \quad \text{and} \quad G(z) = G_1(z) = \frac{1}{\widetilde{q}_1(z) + G_2(z)}.$$

It follows from Theorem 1.18 that the degrees and the leading coefficients of the polynomials q_1 and $\widetilde{q}_1$ coincide. According to (1.51), the functions R_2 and G_2 have equal Hankel minors. So we can apply the same argument to them. Thus, Theorem 1.18 shows that the degrees and the leading coefficients of each pair of the polynomials $q_j, \widetilde{q}_j$ are equal to each other and, consequently, the number of those polynomials must be the same as well. Since the functions R and G have equal number of poles as was proved above, the equality (1.64) thus follows. $\square$

Remark 1.21. Note that the equalities $D_j(R) = D_j(G)$ do not imply the equality of the functions $R \equiv G$. Counterexamples are quite easy to construct. For instance, we can take

$$\begin{aligned} R(z) &= \cfrac{1}{z - 1 - \cfrac{1}{z-2}} = \frac{1}{z} + \frac{1}{z^2} + \frac{2}{z^3} + \frac{5}{z^4} + \cdots\ , \\ G(z) &= \cfrac{1}{z - 1 - \cfrac{1}{z-3}} = \frac{1}{z} + \frac{1}{z^2} + \frac{2}{z^3} + \frac{6}{z^4} + \cdots\ . \end{aligned}$$

Then $R \not\equiv G$ but $D_1(R) = D_2(R) = D_1(G) = D_2(G) = 1$.

Finally, let the function R have a continued fraction expansion (1.46) and also a Laurent series expansion

$$R(z) = \frac{s_{n_1-1}}{z^{n_1}} + \frac{s_{n_1}}{z^{n_1+1}} + \frac{s_{n_1+1}}{z^{n_1+2}} + \cdots\ , \tag{1.66}$$

where $n_1 = \deg q_1 \geq 1$.

Consider the functions

$$F_j(z) := \frac{Q_j(z)}{P_j(z)} = \cfrac{1}{q_1(z) + \cfrac{1}{q_2(z) + \cfrac{1}{q_3(z) + \cfrac{1}{\ddots + \cfrac{1}{q_j(z)}}}}}, \qquad j = 1, \dots, k, \tag{1.67}$$

constructed using the polynomials (1.61) by the Euclidean algorithm (1.42).

Definition 1.22. The polynomials Q_j are called *partial numerators,* the polynomials P_j *partial denominators,* and the fractions F_j *partial quotients* of R.

The denominator $P_j(z)$ of the fraction $F_j(z)$ is the jth leading principal minor of the matrix

$$\begin{pmatrix} q_1(z) & -1 & 0 & \dots & 0 & 0 \\ 1 & q_2(z) & -1 & \dots & 0 & 0 \\ 0 & 1 & q_3(z) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{k-1}(z) & -1 \\ 0 & 0 & 0 & \cdots & 1 & q_k(z) \end{pmatrix},$$

and $P_k(z) = h_0(z)$ (see (1.46)). Let m_j denote the sum of the degrees n_1 through n_j (see (1.61)):

$$m_j = n_1 + n_2 + \dots + n_j, \quad j = 1, 2, \dots, k. \tag{1.68}$$

Then $\deg P_j = m_j$.

Notice that, for a fixed number j $(1 \le j \le k)$, the initial terms of the Laurent series of the function F_j coincide with those in the Laurent series (1.66) of R up to, and including, the term $\frac{s_{2m_j-1}}{z^{2m_j}}$:

$$F_j(z) = \frac{Q_j(z)}{P_j(z)} = \frac{s_{n_1-1}}{z^{n_1}} + \frac{s_{n_1}}{z^{n_1+1}} + \dots + \frac{s_{2m_j-1}}{z^{2m_j}} + \frac{s^{(j)}_{2m_j}}{z^{2m_j+1}} + \cdots \tag{1.69}$$

In fact, each coefficient s_i of the series (1.66) can be found from the recurrence relation

$$s_i = \lim_{z\to\infty} \left[z^{i+1}R(z) - s_{n_1-1}z^{i-n_1+1} - s_{n_1}z^{i+n_1} - \dots - s_{i-1}z \right], \qquad i = n_1 - 1, n_1, n_1 + 1, \dots \tag{1.70}$$

Using the expansions (1.46) and (1.67) of the functions R and F_j, respectively, together with the formula (1.70), we obtain (1.69).

To find an explicit formula for the polynomials P_j that depends only on the coefficients s_i, we introduce the following notation:

$$P_j(z) =: P_{0,j}z^{m_j} + P_{1,j}z^{m_j-1} + \dots + P_{m_j-1,j}z + P_{m_j,j}, \qquad j = 1, 2, \dots, k.$$

For a fixed number j between 1 and k, the formula (1.69) implies

$$Q_j(z) = P_j(z) \left[\frac{s_{n_1-1}}{z^{n_1}} + \frac{s_{n_1}}{z^{n_1+1}} + \dots + \frac{s_{2m_j-1}}{z^{2m_j}} + \frac{s^{(j)}_{2m_j}}{z^{2m_j+1}} + \cdots \right]. \tag{1.71}$$

We must require that the coefficients of $z^{-1}, z^{-2}, \dots, z^{-m_j}$ be zero, which leads to the system

$$\begin{array}{lcl} s_0P_{m_j,j} + s_1P_{m_j-1,j} + \dots + s_{m_j-1}P_{1,j} + s_{m_j}P_{0,j} & = & 0, \\ s_1P_{m_j,j} + s_2P_{m_j-1,j} + \dots + s_{m_j}P_{1,j} + s_{m_j+1}P_{0,j} & = & 0, \\ \vdots & \vdots & \vdots \\ s_{m_j-1}P_{m_j,j} + s_{m_j}P_{m_j-1,j} + \dots + s_{2m_j-2}P_{1,j} + s_{2m_j-1}P_{0,j} & = & 0, \end{array} \tag{1.72}$$

where we set $s_0 := s_1 := \cdots := s_{n_1-2} := 0$. By Cramer's rule, the solution to the system (1.72) satisfies

$$P_{m_j-i,j} = \frac{(-1)^{m_j-i}P_{0,j}}{D_{m_j}(R)} \cdot \begin{vmatrix} s_0 & s_1 & \dots & s_{i-1} & s_{i+1} & \dots & s_{m_j} \\ s_1 & s_2 & \dots & s_i & s_{i+2} & \dots & s_{m_j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{m_j-1} & s_{m_j} & \dots & s_{m_j+i-2} & s_{m_j+i} & \dots & s_{2m_j-1} \end{vmatrix}, \qquad \begin{array}{l} i = 0,1,2,\dots,m_j-1, \\ j = 1,2,\dots,k. \end{array}$$

This formula implies the following representation for the polynomials P_j:

$$P_j(z) = \frac{P_{0,j}}{D_{m_j}(R)} \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{m_j} \\ s_1 & s_2 & s_3 & \dots & s_{m_j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m_j-1} & s_{m_j} & s_{m_j+1} & \dots & s_{2m_j-1} \\ 1 & z & z^2 & \dots & z^{m_j} \end{vmatrix}, \quad j = 1,2,\dots,k. \tag{1.73}$$

Here, according to the notation (1.61),

$$P_{0,j} = \prod_{i=1}^{j} \alpha_i, \quad j = 1,2,\dots,k.$$

This formula, combined with (1.62) and the notation (1.68), yields

$$\begin{aligned} P_{0,2j} &= (-1)^j \prod_{i=1}^{j} \left(\frac{(-1)^{\frac{n_{2i-1}(n_{2i-1}-1)}{2}} D_{m_{2i-1}}(R)}{D_{m_{2i-2}}(R)} \right)^{\frac{1}{n_{2i-1}}} \left(\frac{(-1)^{\frac{n_{2i}(n_{2i}-1)}{2}} D_{m_{2i-1}}(R)}{D_{m_{2i}}(R)} \right)^{\frac{1}{n_{2i}}} \\ &\quad \text{for} \quad j = 1,2,\dots,\left\lfloor \frac{k}{2} \right\rfloor. \\ P_{0,2j+1} &= (-1)^j \prod_{i=1}^{j} \left(\frac{(-1)^{\frac{n_{2i-1}(n_{2i-1}-1)}{2}} D_{m_{2i-2}}(R)}{D_{m_{2i-1}}(R)} \right)^{\frac{1}{n_{2i-1}}} \left(\frac{(-1)^{\frac{n_{2i}(n_{2i}-1)}{2}} D_{m_{2i}}(R)}{D_{m_{2i-1}}(R)} \right)^{\frac{1}{n_{2i}}} \times \\ &\quad \times \left(\frac{(-1)^{\frac{n_{2j+1}(n_{2j+1}-1)}{2}} D_{m_{2j}}(R)}{D_{m_{2j+1}}(R)} \right)^{\frac{1}{n_{2i+1}}} \qquad \text{for} \quad j = 0,1,2,\dots,\left\lceil \frac{k}{2} \right\rceil - 1. \end{aligned}$$

In case the Euclidean algorithm is *regular,* the notion that will be introduced in the sequel, these formulæ take a simpler form.

The coefficients of Q_j can be obtained from the coefficients of P_j via the formula (1.71): denoting

$$Q_j(z) = Q_{0,j}z^{m_j-n_1} + Q_{1,j}z^{m_j-n_1-1} + \cdots + Q_{m_j-n_1-1,j}z + Q_{m_j-n_1,j},$$

we find the coefficients of Q_j by the following

$$Q_{i,j} = \frac{(-1)^{m_j}P_{0,j}}{D_{m_j}(R)} \cdot \begin{vmatrix} 0 & 0 & \dots & s_0 & s_1 & \dots & s_{n_1+i-1} \\ s_0 & s_1 & \dots & s_{m_j-n_1-i+1} & s_{m_j-n_1-i+2} & \dots & s_{m_j} \\ s_1 & s_2 & \dots & s_{m_j-n_1-i+2} & s_{m_j-n_1-i+3} & \dots & s_{m_j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{m_j-1} & s_{m_j} & \dots & s_{2m_j-n_1-i} & s_{2m_j-n_1-i+1} & \dots & s_{2m_j-1} \end{vmatrix}$$

for $i = 0,1,\dots,m_j-n_1$.

Note that the rational functions $F_j(z) = \dfrac{Q_j(z)}{P_j(z)}$ are exactly the diagonal elements of the Padé table of the function R, see [10, 43].

1.4 Euclidean algorithm: regular case. Finite continued fractions of Jacobi type

In this section we discuss the best known case of the Euclidean algorithm, which leads to orthogonal polynomials, 3-term recurrence relations and other related phenomena. We give conditions for regularity and discuss the form of partial quotients and the generalized eigenvalue problem corresponding to the regular case.

Suppose that all the polynomials q_j resulting from an application the Euclidean algorithm (1.42) are linear:

$$q_j(z) = \alpha_j z + \beta_j, \quad \alpha_j, \beta_j \in \mathbb{C}, \qquad \alpha_j \neq 0, \quad j = 1, \dots, r. \tag{1.74}$$

We call this situation the *regular case* of the Euclidean algorithm. In the regular case,

$$\deg f_j(z) = n - j, \qquad j = 0, \dots, r, \tag{1.75}$$

and $r = n - l$, where l is the degree of the greatest common divisor f_r of the polynomials f_0 and f_1 (and of all other polynomials f_j in the sequence).

Thus, the polynomials f_j satisfy the following *three-terms recurrence relation*:

$$f_{j-1}(z) = (\alpha_j z + \beta_j) f_j(z) + f_{j+1}(z), \quad j = 1, \dots, r. \tag{1.76}$$

Consequently, the function R expands into the continued fraction

$$R(z) = \frac{f_1(z)}{f_0(z)} = \frac{h_1(z)}{h_0(z)} = \cfrac{1}{\alpha_1 z + \beta_1 + \cfrac{1}{\alpha_2 z + \beta_2 + \cfrac{1}{\alpha_3 z + \beta_3 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_r z + \beta_r}}}}}, \tag{1.77}$$

where the polynomials h_0 and h_1 are defined by (1.43).

Definition 1.23. Continued fractions of the type (1.77) are usually called *J-fractions* or continued fractions of Jacobi type.

Remark 1.24. If a rational function G satisfies the condition $\lim\limits_{z\to\infty} G(z) = c$, $0 < |c| < \infty$, then we will say that $G(z)$ has a J-fraction expansion if the function $R(z) = G(z) - \lim\limits_{z\to\infty} G(z)$ can be represented as in (1.77).

Theorem 1.18 with some simple modifications implies the following corollary, which will be useful later.

Corollary 1.25. *Two rational functions R and G, both vanishing at ∞, satisfy the following condition*

$$R(z) = \frac{1}{\alpha z + \beta + G(z)}, \quad \alpha \neq 0,$$

if and only if

$$D_j(R) = \frac{(-1)^{j-1}}{\alpha^{2j-1}} D_{j-1}(G), \quad j = 1, 2, \dots, \tag{1.78}$$

where $D_0(G) := 1$.

Using Corollary 1.25, we can now prove a criterion when a rational function expands into a J-fraction.

Theorem 1.26 ([74])**.** *A rational function*

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots \tag{1.79}$$

with exactly r poles has a J-fraction expansion if and only if

$$D_j(R) \neq 0, \quad j = 1, \dots, r, \tag{1.80}$$

where $D_j(R)$ are the Hankel determinants defined in (1.6).

Proof. In view of Remark 1.24 it is sufficient to consider the case $s_{-1} = 0$. Suppose that the function R has a J-fraction expansion (1.77), i.e., a continued fraction expansion (1.46) with polynomials q_j satisfying (1.74) with $k = r$. Then the formulæ (1.62) yield

$$D_j(R) = (-1)^{\frac{j(j-1)}{2}} \prod_{i=1}^{j} \frac{1}{\alpha_i^{2j-2i+1}} \neq 0, \quad j = 1, 2, \ldots, r. \tag{1.81}$$

Now suppose that the inequalities (1.80) hold. Then $D_1(R_1) = s_0^{(1)} \neq 0$, where we denote[6]

$$R_1(z) := R(z) = \frac{s_0^{(1)}}{z} + \frac{s_1^{(1)}}{z^2} + \frac{s_2^{(1)}}{z^3} + \cdots$$

Therefore, $R_1(z)$ can be represented as follows

$$R_1(z) = \frac{1}{\alpha_1 z + \beta_1 + R_2(z)}, \quad \alpha_1 = \frac{1}{s_0^{(1)}} \neq 0,$$

Now, Corollary 1.25 implies that $D_j(R_1) = \dfrac{(-1)^{j-1}}{\alpha_1^{2j-1}} D_{j-1}(R_2)$, $j = 1, 2, \ldots$. Therefore, the function R_2 is of the same type as R_1, that is, R_1 satisfies the conditions[7]

$$D_j(R_2) \neq 0, \quad j = 1, 2, \ldots, r-1,$$

which are analogous to (1.80). In particular, $s_0^{(2)} \neq 0$, where

$$R_2(z) = \frac{s_0^{(2)}}{z} + \frac{s_1^{(2)}}{z^2} + \frac{s_2^{(2)}}{z^3} + \cdots$$

If we continue this process, then for each function

$$R_j(z) = \frac{s_0^{(j)}}{z} + \frac{s_1^{(j)}}{z^2} + \frac{s_2^{(j)}}{z^3} + \cdots$$

we obtain that $s_0^{(j)} \neq 0$, $j = 1, 2, \ldots, r$. Consequently,

$$R_j(z) = \frac{1}{\alpha_j z + \beta_j + R_{j+1}(z)}, \quad j = 1, 2, \ldots, r,$$

where $R_{r+1} \equiv 0$. This means that the function R can be represented as a J-fraction (see (1.45)–(1.46)). $\square$

This theorem and Theorem 1.5 imply the following statement, which was proved in [101] (Theorem 41.1) with inessential differences.

Corollary 1.27. *Let $R(z) = \dfrac{q(z)}{p(z)}$, where p and q are defined in* (1.2)–(1.3)*, and let the function R have exactly r poles. The function R can be expanded into a J-fraction* (1.77) *if and only if*

$$\nabla_{2j}(p, q) \neq 0, \qquad j = 1, \ldots, r, \tag{1.82}$$

where $\nabla_{2j}(p, q)$ are defined in (1.12).

Corollary 1.28. *Given a rational function $R(z) = \dfrac{f_1(z)}{f_0(z)}$, where $\deg f_1 < \deg f_0$, the Euclidean algorithm* (1.42) *applied to the polynomials f_0 and f_1 is regular if and only if the inequalities* (1.80) *hold.*

[6] Recall that we assumed $s_{-1} = 0$.

[7] For $j \geq r$, we have $D_{j+1}(R_1) = D_j(R_2) = 0$, and therefore R_2 has exactly $r - 1$ poles.

If we make an equivalence transformation of the fraction (1.77), as in [101, p. 166], and set

$$d_0 := \frac{1}{\alpha_1}, \quad d_j := -\frac{1}{\alpha_j\alpha_{j+1}}, \qquad j = 1, 2, \dots, r-1. \tag{1.83}$$

$$e_j := -\frac{\beta_j}{\alpha_j}, \qquad j = 1, 2, \dots, r, \tag{1.84}$$

then the J-fraction (1.77) takes the form

$$R(z) = \cfrac{d_0}{z - e_1 - \cfrac{d_1}{z - e_2 - \cfrac{d_2}{\ddots - \cfrac{d_{r-1}}{z - e_r}}}}. \tag{1.85}$$

Moreover, from (1.81) and (1.83) we have (see [101, p. 167])

$$D_j(R) = D_{j-1}(R) \cdot \prod_{i=1}^{j} d_{i-1}. \tag{1.86}$$

In fact,

$$\begin{aligned} D_j(R) &= (-1)^{\frac{j(j-1)}{2}} \prod_{i=1}^{j} \frac{1}{\alpha_i^{2j-2i+1}} = (-1)^{\frac{j(j-1)}{2}} \frac{1}{\alpha_j} \cdot \prod_{i=1}^{j-1} \frac{1}{\alpha_i^2} \cdot \prod_{i=1}^{j-1} \frac{1}{\alpha_i^{2j-2i-1}} = \\ &= (-1)^{\frac{j(j-1)}{2}} (-1)^{\frac{(j-2)(j-1)}{2}} (-1)^{j-1} \frac{1}{\alpha_1} \cdot \prod_{i=1}^{j-1} \frac{-1}{\alpha_i\alpha_{i+1}} \cdot D_{j-1}(R) = D_{j-1}(R) \cdot \prod_{i=1}^{j} d_{i-1}. \end{aligned}$$

The expression (1.86) implies an interesting formula:

$$d_j = \frac{D_{j-1}(R) D_{j+1}(R)}{D_j^2(R)}, \qquad j = 0, 1, \dots, r-1, \tag{1.87}$$

where we set $D_{-1}(R) := D_0(R) := 1$.

Remark 1.29. From the formulæ (1.83), (1.84) and (1.87) it follows that both forms of J-fraction expansions (1.77) and (1.85) of the function $R(z)$ are *unique*, that is, all coefficients in (1.77) and (1.85) are defined uniquely for the function R.

Remark 1.30. (cf. [101, p.170]) Suppose that the function R has a J-fraction expansion (1.77). Then

$$R(-z) = -\cfrac{1}{\alpha_1 z - \beta_1 + \cfrac{1}{\alpha_2 z - \beta_2 + \cfrac{1}{\alpha_3 z - \beta_3 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_r z - \beta_r}}}}}. \tag{1.88}$$

According to Remark 1.29, (a properly normalized) J-fraction expansion is unique. Therefore, if the function R is odd, i.e., $R(z) \equiv -R(-z)$, then all β_j in (1.88) must be equal to zero. Conversely, if β_j are all equal to zero, then $R(z)$ is obviously an odd function of z.

Thus, the formula (1.86) implies

$$D_j(R) = \prod_{i=0}^{j} d_i^{j-i} \tag{1.89}$$

In other words, if the function R has a J-fraction expansion (1.85), the minors $D_j(R)$ do not depend on the coefficients e_j, $j = 1, \dots, r$, whereas the minors $\widehat{D}_j(R)$ obviously do. In order to establish the dependence of these minors on the coefficients of the fraction (1.85), we first prove the following simple fact.

Lemma 1.31. *Let the complex rational function*

$$R(z)=\frac{s_0}{z}+\frac{s_1}{z^2}+\frac{s_2}{z^3}+\cdots \tag{1.90}$$

with exactly r poles, has a J-fraction expansion (1.85). *Then*

$$\frac{\widehat{D}_r(R)}{D_r(R)}=\prod_{k=1}^{r}\mu_k, \tag{1.91}$$

where μ_k are the poles of the function R, and the minors $D_j(R)$ and $\widehat{D}_j(R)$, $j=1,2,\ldots$, are defined by (1.6) *and* (1.9).

Proof. If R has a pole at zero, then the lemma holds true, since in this case $D_r(R)\neq 0$ by Theorem 1.26, while $\widehat{D}_r(R)=0$ by Corollary 1.4. Now assume that R has no a pole at zero.

At first, suppose that the function R has only simple poles $\mu_k\neq\mu_j$ whenever $k\neq j$, so it can be represented as a sum of partial fractions

$$R(z)=\sum_{k=1}^{r}\frac{\nu_k}{z-\mu_k}$$

This formula together with (1.90) gives the following well-known formulæ

$$s_j=\sum_{k=1}^{r}\nu_k\mu_k^j,\qquad j=0,1,2,\ldots \tag{1.92}$$

On the other hand, (1.22) implies

$$D_r(G)=\prod_{k=1}^{r}\nu_k\cdot\begin{vmatrix}1&1&\ldots&1\\ \mu_1&\mu_2&\ldots&\mu_r\\ \vdots&\vdots&\ddots&\vdots\\ \mu_1^{r-1}&\mu_2^{r-1}&\ldots&\mu_r^{r-1}\end{vmatrix}^2. \tag{1.93}$$

Further, recall that $\widehat{D}_r(R)=D_r(\Phi)$, where

$$\Phi(z)=zR(z)=\sum_{k=1}^{r}\nu_k+\sum_{k=1}^{r}\frac{\nu_k\mu_k}{z-\mu_k}=s_0+\frac{s_1}{z}+\frac{s_2}{z^2}+\frac{s_3}{z^3}+\ldots$$

So, analogously to (1.93), we obtain

$$\widehat{D}_r(R)=D_r(\Phi)=\prod_{k=1}^{r}(\nu_k\mu_k)\cdot\begin{vmatrix}1&1&\ldots&1\\ \mu_1&\mu_2&\ldots&\mu_r\\ \vdots&\vdots&\ddots&\vdots\\ \mu_1^{r-1}&\mu_2^{r-1}&\ldots&\mu_r^{r-1}\end{vmatrix}^2. \tag{1.94}$$

Now (1.91) follows from (1.93) and (1.94).

Let $R=\dfrac{q}{p}$ and suppose that the function R has multiple poles, or, equivalently, that the polynomial p has multiple zeros. In this case, we can consider an approximating polynomial p_ε with simple zeros such that

$$\lim_{\varepsilon\to 0}p_\varepsilon(z)=p(z)\quad\text{for all}\ \ z.$$

Then

$$D_r(R_\varepsilon)\xrightarrow[\varepsilon\to 0]{}D_r(R),\qquad \widehat{D}_r(R_\varepsilon)\xrightarrow[\varepsilon\to 0]{}\widehat{D}_r(R),$$

where $R_\varepsilon(z):=\dfrac{q(z)}{p_\varepsilon(z)}$. The formula (1.91) is valid for the polynomials q and p_ε, so it is also valid at the limit, i.e., for the polynomials q and p, since the product of all zeros of the polynomial $p_\varepsilon(z)$ tends to the product of all zeros of the polynomial p whenever $\varepsilon\to 0$. $\square$

Let us consider the following tridiagonal matrix

$$\mathcal{J}_r = \begin{pmatrix} e_1 & \sqrt{d_1} & 0 & \dots & 0 & 0 \\ \sqrt{d_1} & e_2 & \sqrt{d_2} & \dots & 0 & 0 \\ 0 & \sqrt{d_2} & e_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_{r-1} & \sqrt{d_{r-1}} \\ 0 & 0 & 0 & \dots & \sqrt{d_{r-1}} & e_r \end{pmatrix} \tag{1.95}$$

constructed using the coefficients of the J-fraction (1.85).

Suppose that the coefficient d_0 in (1.85) equals 1 and consider the partial quotients of the J-fraction (1.85)

$$F_j(z) = \frac{Q_j(z)}{P_j(z)} = \cfrac{1}{z - e_1 - \cfrac{d_1}{z - e_2 - \cfrac{d_2}{\ddots - \cfrac{d_{j-1}}{z - e_j}}}}, \qquad j = 1, \dots, r. \tag{1.96}$$

Then $F_r = R$ and the polynomial P_j is the characteristic polynomial of the leading principal submatrix of $\mathcal{J}_r$ of order j, for any $j = 1, \dots, r$.

Using the formula (1.91), it is now easy to prove the following theorem.

Theorem 1.32. *Let the matrix J_n be defined by* (1.95) *and let the function R defined by* (1.90) *have a J-fraction expansion* (1.85) *(with $d_0 = 1$). Then the leading principal minors $|\mathcal{J}_r|_1^m$, $m = 1, \dots, r$, of the matrix $\mathcal{J}_r$ can be found by the following formulæ:*

$$|\mathcal{J}_r|_1^m = \frac{\widehat{D}_m(R)}{D_m(R)}, \qquad m = 1, \dots, r. \tag{1.97}$$

Proof. At first, we note that the formula (1.97) holds for $\det(\mathcal{J}_r) = |\mathcal{J}_r|_1^r$. Indeed, on the one hand, the determinant of the matrix $\mathcal{J}_r$ is the product of its eigenvalues. On the other hand, the eigenvalues of the matrix $\mathcal{J}_r$ are the poles of the function R, so by (1.91), their product equals $\dfrac{\widehat{D}_r(R)}{D_r(R)}$.

Let us now turn to the functions F_j introduced in (1.96). According to (1.69), we have

$$F_m(z) = \frac{Q_m(z)}{P_m(z)} = \frac{s_0}{z} + \frac{s_1}{z^2} + \dots + \frac{s_{2m-1}}{z^{2m}} + \frac{s_{2m}^{(m)}}{z^{2m+1}} + \cdots, \qquad m = 1, \dots, r, \tag{1.98}$$

where the coefficients s_i, $i = 0, 1, \dots, 2m-1$ coincide with those of the function R defined in (1.90). Consequently,

$$D_j(F_m) = D_j(R), \qquad \widehat{D}_j(F_m) = \widehat{D}_j(R), \qquad j = 1, \dots, m.$$

Thus, applying Lemma 1.31 to the function F_m and to the leading principal submatrix of $\mathcal{J}_r$ of order m, we obtain

$$|\mathcal{J}_r|_1^m = \frac{\widehat{D}_j(F_m)}{D_j(F_m)} = \frac{\widehat{D}_m(R)}{D_m(R)}, \qquad m = 1, \dots, n,$$

as required. $\square$

Now suppose that the function R has a J-fraction expansion (1.85), where d_0 may differ from 1. Then we can consider the function $G(z) := \dfrac{R(z)}{d_0}$. It is clear that

$$D_m(R) = d_0^m D_m(G), \qquad \widehat{D}_j(R) = d_0^m \widehat{D}_j(G) \qquad m = 1, \dots, r.$$

Consequently, according to (1.97), we get

$$|\mathcal{J}_r|_1^m = \frac{\widehat{D}_m(G)}{D_m(G)} = \frac{\widehat{D}_m(R)}{D_m(R)}, \qquad m = 1, \dots, r. \tag{1.99}$$

Thus, if the function R has a J-fraction expansion (1.85), then it follows from (1.99) and (1.97) that the minors $\widehat{D}_j(R)$ can be found as follows

$$\widehat{D}_m(R) = |\mathcal{J}_r|_1^m \cdot \prod_{i=0}^{m-1} d_i^{m-i}, \qquad m = 1, \dots, r,$$

where $|\mathcal{J}_r|_1^m$ is the leading principal minor of order m of the matrix $\mathcal{J}_r$, $m = 1, \dots, r$.

Finally, note that the matrix $\mathcal{J}_j$ can be replaced throughtout the discussion above by the matrix

$$\begin{pmatrix} e_1 & 1 & 0 & \dots & 0 & 0 \\ d_1 & e_2 & 1 & \dots & 0 & 0 \\ 0 & d_2 & e_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_{r-1} & 1 \\ 0 & 0 & 0 & \dots & d_{r-1} & e_r \end{pmatrix}$$

which is diagonally similar to the matrix $\mathcal{J}_r$ via the transformation

$$\operatorname{diag}(1, \sqrt{d_1}, \sqrt{d_1 d_2}, \dots, \sqrt{d_1, \dots, d_{r-1}}),$$

which also preserves all principal minors.

Let us now return to the equation (1.48). If all the polynomials q_i in the fraction (1.46) are linear, as in (1.74), then the equation (1.48) becomes a generalized eigenvalue problem

$$(\mathrm{A}z + \mathrm{B})u = 0, \tag{1.100}$$

for the matrix pair (A, B) of order r where A is diagonal and B is tridiagonal (for a similar setup, see [38]):

$$\mathrm{A} = \begin{pmatrix} \alpha_r & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha_{r-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha_{r-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & \alpha_1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} \beta_r & -1 & 0 & \dots & 0 & 0 \\ 1 & \beta_{r-1} & -1 & \dots & 0 & 0 \\ 0 & 1 & \beta_{r-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_2 & -1 \\ 0 & 0 & 0 & \dots & 1 & \beta_1 \end{pmatrix}, \tag{1.101}$$

and the function R expands into a J-fraction (1.77).

The polynomials h_j ($j = 0, \dots, r-1$) in (1.77) are known to be the leading principal minors of order $r-j$ of the matrix $\mathcal{J}(z) = z\mathrm{A}+\mathrm{B}$ as we mentioned above (see also [11, 34]). In particular, $h_0(z) = \det(z\mathrm{A}+\mathrm{B})$.

As we already mentioned in Section 1.3, we can localize the eigenvalues of the problem (1.100) using properties of the function R. For example, if $\alpha_j > 0, \beta_j > 0$ for all $j = 1, \dots, r$ in (1.74) (or, equivalently, in (1.77)), then every polynomial q_j is a function mapping the closed right half-plane into the open right half-plane. In fact, $\operatorname{Re} q_j(z) = \alpha_j \operatorname{Re} z + \beta_j > 0$, whenever $\operatorname{Re} z \geq 0$. Now note: if functions F_1 and F_2 map the closed right-half plane to the open right-half plane, then so do the functions $F_1 + F_2$ and $\dfrac{1}{F_j}$ $(j = 1, 2)$. Consequently, in this case the function R represented by (1.77) also maps the closed right half-plane to the open right half-plane, being a composition of such maps. But since the function R has a positive real part in the closed right half-plane and is finite there, then all its zeros and poles and, subsequently, all zeros of the polynomials h_0 and h_1 lie in the open left half-plane. In summary, *if* $\alpha_j, \beta_j > 0$, $j = 1, 2 \dots, r$, *then the eigenvalue problem* (1.100) *is stable*, that is, *all its eigenvalues lie in the open left half-plane.* This example constitutes the subject of Problem 7.1 in [11]. A similar result was obtained in [37].

Finally, at the end of this subsection, let us discuss the form taken by the partial quotients F_j defined by (1.67) in the regular case of the Euclidean algorithm. Since all polynomials q_j are of degree one (see (1.74)), we have $k = r$, the number of poles of R. Moreover, for a fixed integer j $(1 \leq j \leq r)$ we have

$\deg P_j = m_j = j$, where P_j is denominator of the fraction F_j of the form (cf. (1.73))

$$P_j(z) = \frac{P_{0,j}}{D_j(R)} \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_j \\ s_1 & s_2 & s_3 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \dots & s_{2j-1} \\ 1 & z & z^2 & \dots & z^j \end{vmatrix}, \qquad j = 1, 2, \dots, r.$$

The leading coefficients $P_{0,j}$ can be determined by the formulæ (see, e.g., [67])

$$\begin{aligned} P_{0,2j} &= (-1)^j \frac{D_1^2(R)D_3^2(R)\cdots D_{2j-1}^2(R)}{D_2^2(R)D_4^2(R)\cdots D_{2j-2}^2(R)D_{2j}(R)}, \qquad j = 1, 2, \dots, \left\lfloor \frac{r}{2} \right\rfloor. \\ P_{0,2j+1} &= (-1)^j \frac{D_2^2(R)D_4^2(R)\cdots D_{2j}^2(R)}{D_1^2(R)D_3^2(R)\cdots D_{2j-1}^2(R)D_{2j+1}(R)}, \quad j = 0, 1, 2, \dots, \left\lceil \frac{r}{2} \right\rceil - 1. \end{aligned}$$

1.5 Euclidean algorithm: doubly regular case. Finite continued fractions of Stieltjes type

Assume, as above, that the rational function R has a series expansion (1.79), where $s_{-1} = 0$, and consider the function

$$F(z) := zR(z^2) = \frac{s_0}{z} + \frac{s_1}{z^3} + \frac{s_2}{z^5} + \cdots \tag{1.102}$$

The function F can be also represented as a series

$$F(z) = \frac{t_0}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \cdots, \tag{1.103}$$

where

$$\begin{aligned} t_{2j} &= s_j, \\ t_{2j+1} &= 0, \end{aligned} \qquad j = 0, 1, \dots \tag{1.104}$$

Remark 1.33. Let r denote the number of poles of the function R (counting multiplicities). Note that the function F has $2r$ poles if and only if R has no pole at zero. Otherwise, F has only $2r-1$ poles.

Lemma 1.34. *The following relations hold between the minors $D_j(R)$, $\widehat{D}_j(R)$ and $D_j(F)$ defined in* (1.6) *and* (1.9)*:*

$$\begin{aligned} D_{2j}(F) &= D_j(R) \cdot \widehat{D}_j(R), \\ D_{2j-1}(F) &= D_j(R) \cdot \widehat{D}_{j-1}(R), \end{aligned} \qquad j = 1, 2, \dots, \tag{1.105}$$

where we set $\widehat{D}_0(R) := 1$.

Proof. First interchange the rows and columns of the determinant $D_{2j}(F)$

$$D_{2j}(F) = \begin{vmatrix} t_0 & t_1 & \dots & t_{2j-2} & t_{2j-1} \\ t_1 & t_2 & \dots & t_{2j-1} & t_{2j} \\ t_2 & t_3 & \dots & t_{2j} & t_{2j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{2j-3} & t_{2j-2} & \dots & t_{4j-4} & t_{4j-3} \\ t_{2j-2} & t_{2j-1} & \dots & t_{4j-3} & t_{4j-2} \\ t_{2j-1} & t_{2j} & \dots & t_{4j-2} & t_{4j-2} \end{vmatrix} = \begin{vmatrix} s_0 & 0 & s_1 & \dots & s_{j-1} & 0 \\ 0 & s_1 & 0 & \dots & 0 & s_j \\ s_1 & 0 & s_2 & \dots & s_j & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & s_{j-1} & 0 & \dots & 0 & s_{2j-2} \\ s_{j-1} & 0 & s_j & \dots & s_{2j-2} & 0 \\ 0 & s_j & 0 & \dots & 0 & s_{2j-1} \end{vmatrix}$$

so that $(2i{-}1)$st row and $(2i{-}1)$st column move to the ith position, for each $i = 2, 3, \dots, j$; this produces

$$D_{2j}(F) = \begin{vmatrix} s_0 & s_1 & \dots & s_{j-1} & 0 & 0 & \dots & 0 \\ s_1 & s_2 & \dots & s_j & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & \dots & s_{2j-2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & s_1 & s_2 & \dots & s_j \\ 0 & 0 & \dots & 0 & s_2 & s_3 & \dots & s_{j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & s_j & s_{j+1} & \dots & s_{2j-1} \end{vmatrix} = D_j(R) \cdot \widehat{D}_j(R).$$

The formula for the determinants $D_{2j-1}(F)$ can be proved in the same way. ☐

This lemma, Theorem 1.26 and Corollary 1.4 can now be combined to derive the following corollary, which will be important later.

Corollary 1.35. *The function F defined in* (1.102) *has a J-fraction expansion if and only if*

$$D_j(R) \neq 0, \qquad j = 1, 2, \dots, r, \tag{1.106}$$

$$\widehat{D}_j(R) \neq 0, \qquad j = 1, 2, \dots, r{-}1, \tag{1.107}$$

$$D_j(R) = \widehat{D}_j(R) = 0, \quad j = r{+}1,\ r{+}2, \dots \tag{1.108}$$

where r is the number of the poles of the function R, which generates F. In addition, $\widehat{D}_r(R) = 0$ if and only if the function R (hence also the function F) has a pole at 0.

Since F is evidently an odd function, Remark 1.30 shows that its J-fraction expansion (if any) has to be of the following form:

$$F(z) = \cfrac{1}{c_1 z + \cfrac{1}{c_2 z + \cfrac{1}{c_3 z + \cfrac{1}{\ddots + \cfrac{1}{c_k z}}}}}, \quad c_j \neq 0, \quad j = 1, 2, \dots, k. \tag{1.109}$$

Here $k = 2r$ if the function R has no pole at 0 and $k = 2r - 1$ if it does, as follows from Remark 1.33 and Theorem 1.26. Making the equivalence transformation (1.83)–(1.84) in (1.109) with c_i replacing α_i and with $e_i = \beta_i = 0$, we obtain by induction

$$c_{2j} = -\frac{\prod_{i=0}^{j-1} d_{2i}}{\prod_{i=0}^{j} d_{2i-1}}, \qquad j = 1, 2, \dots, r, \tag{1.110}$$

$$c_1 = \frac{1}{d_0}, \qquad c_{2j-1} = -\frac{\prod_{i=0}^{j-1} d_{2i-1}}{\prod_{i=0}^{j-1} d_{2i}}, \qquad j = 2, 3, \dots, r, \tag{1.111}$$

since $c_{j+1} = -\dfrac{1}{c_j d_j}$ from (1.83). The formulæ (1.87) and (1.105) imply

$$d_{2i} = \frac{D_{2i-1}(F) \cdot D_{2i+1}(F)}{D_{2i}^2(F)} = \frac{D_{i+1}(R) \cdot \widehat{D}_{i-1}(R)}{D_i(R) \cdot \widehat{D}_i(R)}, \qquad i = 1, 2, \dots, r, \tag{1.112}$$

$$d_{2i-1} = \frac{D_{2i-2}(F) \cdot D_{2i}(F)}{D_{2i-1}^2(F)} = \frac{D_{i-1}(R) \cdot \widehat{D}_i(R)}{D_i(R) \cdot \widehat{D}_{i-1}(R)}, \qquad i = 1, 2, \dots, r, \tag{1.113}$$

Combining (1.110)–(1.111) and (1.112)–(1.113), we get very interesting and ultimately very useful relations between the coefficients c_j and the minors $D_j(R)$ and $\widehat{D}_j(R)$ (see [62]):

$$c_{2j} = -\frac{D_j^2(R)}{\widehat{D}_{j-1}(R)\cdot\widehat{D}_j(R)}, \qquad j = 1,2,\dots,r, \tag{1.114}$$

$$c_{2j-1} = \frac{\widehat{D}_{j-1}^2(R)}{D_{j-1}(R)\cdot D_j(R)}, \qquad j = 1,2,\dots,r. \tag{1.115}$$

Remark 1.36. According to Corollary 1.4, the function R has a pole at zero if and only if $\widehat{D}_{r-1}(R) \neq 0$ and $\widehat{D}_r(R) = 0$. From (1.114) and (1.107) we conclude that $c_{2r} = \infty$ in this case.

Upon making an equivalence transformation in (1.109), replacing z^2 by z, and removing the factor z (as in [101, p. 170]), we obtain

$$R(z) = \cfrac{1}{c_1 z + \cfrac{1}{c_2 + \cfrac{1}{c_3 z + \cfrac{1}{\ddots + \cfrac{1}{T}}}}}, \quad c_j \neq 0, \quad \text{where} \quad T = \begin{cases} c_{2r} & \text{if } |R(0)| < \infty, \\ c_{2r-1}z & \text{if } R(0) = \infty. \end{cases} \tag{1.116}$$

Definition 1.37. Continued fractions of type (1.116) are called *continued fractions of Stieltjes type* or *Stieltjes continued fraction*. Accordingly, if (1.116) holds for a function R with r poles, then we say that R has a Stieltjes continued fraction expansion.

Remark 1.38. If $R(\infty) = c_0$, where $0 < |c_0| < \infty$, then we say that R has a Stieltjes continued fraction expansion whenever the function $G(z) := R(z) - c_0$ has one.

Summarizing all the previous results, we obtain the following criterion for a rational function to have a Stieltjes continued fraction expansion.

Theorem 1.39 ([93, 94, 95, 96, 101, 62])**.** *Suppose that a rational function R is finite at infinity, has exactly r poles, and can be represented as a series* (1.79)*. The function R has a Stieltjes continued fraction expansion* (1.116) *if and only if it satisfies the conditions* (1.106)–(1.108)*. In that case, the coefficients of the Stieltjes continued fraction can be found from the formulæ* (1.114)–(1.115)*.*

Assume that

$$F(z) = zR(z^2) = \frac{g_1(z)}{g_0(z)}, \tag{1.117}$$

and $F(\infty) = 0$. The function F has a J-fraction expansion (1.109) if and only if the Euclidean algorithm applied to the polynomials g_0 and g_1 is regular. Since F is an odd function, (1.109) implies

$$g_{j-1}(z) = c_j z g_j(z) + g_{j+1}(z), \quad c_j \neq 0, \quad j = 1,2,\dots,k,$$

where k is equal to $2r-1$ or $2r$ depending on whether or not R has a pole at zero. Performing the transformation

$$\widetilde{g}_{2i}(z) = g_{2i}(z), \qquad \widetilde{g}_{2i-1}(z) = \frac{g_{2i-1}(z)}{z}, \qquad i = 1,2,\dots,r,$$

we obtain

$$\widetilde{g}_{j-1}(z) = \widetilde{q}_j(z)\widetilde{g}_j(z) + \widetilde{g}_{j+1}(z), \quad q_j(z) \not\equiv 0, \quad j = 1,2,\dots,k,$$

where

$$\begin{aligned} \widetilde{q}_{2i}(z) &= c_{2i}, & i = 1,2,\dots,\left\lfloor\frac{k}{2}\right\rfloor, \\ \widetilde{q}_{2i-1}(z) &= c_{2i-1}z^2, & i = 1,2,\dots,r. \end{aligned} \tag{1.118}$$

If R and, subsequently, F have a pole at zero, then $\widetilde{q}_{2r}(z)$ does not exist, according to Remark 1.36.

Since $F(z) = \dfrac{g_1(z)}{g_0(z)}$ is an odd function (see (1.117)), the function $\dfrac{\widetilde{g}_1(z)}{\widetilde{g}_0(z)} = \dfrac{F(z)}{z}$ is even. Therefore, both polynomials $\widetilde{g}_0$ and $\widetilde{g}_1$ are even, hence so are all subsequent polynomials $\widetilde{g}_j(z)$, $j = 2, \ldots, k$. Equivalently,

$$\widetilde{g}_j(z) = f_j(z^2).$$

From (1.118) we see that the polynomials $\widetilde{q}_j(z)$ are even, so are functions of z^2. Denoting $q_j(z^2) = \widetilde{q}_j(z)$, we obtain

$$f_{j-1}(z^2) = q_j(z^2) f_j(z^2) + f_{j+1}(z^2), \qquad j = 1, 2, \ldots, k.$$

Since $\dfrac{f_1(z^2)}{f_0(z^2)} = \dfrac{\widetilde{g}_1(z)}{\widetilde{g}_0(z)} = \dfrac{F(z)}{z} = R(z^2)$, replacing z^2 by z, we get

$$R(z) = \frac{f_1(z)}{f_0(z)}$$

and

$$f_{j-1}(z) = q_j(z) f_j(z) + f_{j+1}(z), \quad q_j(z) \not\equiv 0, \quad j = 1, 2, \ldots, k, \tag{1.119}$$

and

$$\begin{aligned} q_{2i}(z) &= c_{2i}, && i = 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor, \\ q_{2i-1}(z) &= c_{2i-1} z, && i = 1, 2, \ldots, r. \end{aligned} \tag{1.120}$$

Recalling that we consider the case $R(\infty) = 0$, we see that the polynomials f_j have fixed degrees

$$\begin{aligned} f_{2i}(z) &= h_{2i} z^{n-i} + \cdots, && i = 0, 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor, \\ f_{2i-1}(z) &= h_{2i-1} z^{n-i} + \cdots, && i = 1, 2, \ldots, r. \end{aligned} \tag{1.121}$$

where $n = \deg f_0 \geq r$ and $h_j \neq 0$ for $j = 0, 1, \ldots, k$.

So, the function $R(z) = \dfrac{f_1(z)}{f_0(z)}$ has a Stieltjes continued fraction expansion (1.116) if and only if the application of the Euclidean algorithm to the polynomials f_0 and f_1 has the form (1.119)–(1.120) produces their the greatest common divisor $f_k(z)$, where $k = 2r$, if $|R(0)| < \infty$, and $k = 2r - 1$ otherwise. We already know from the condition (1.106) that the Euclidean algorithm (1.119)–(1.120) must be regular for the function R to have the expansion (1.116). But in this case we obtain one more set of inequalities, namely, (1.107). This justified calling such an instance of the algorithm *doubly regular*.

Now consider again the rational function

$$R(z) = \frac{q(z)}{p(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots, \tag{1.122}$$

where the polynomials p and q are defined in (1.2)–(1.3).

We now introduce another (infinite) matrix associated with the function (1.122). This object differs significantly from the Hankel matrix constructed in (1.5) from the coefficients s_j. In particular, this new matrix is made of the coefficients of the polynomials p and q.

Definition 1.40. Given polynomials p and q from (1.2)–(1.3), define the infinite matrix $H(p, q)$ as follows: if $\deg q < \deg p$, that is, if $b_0 = 0$, then[8]

$$H(p,q) := \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}; \tag{1.123}$$

[8]Generally speaking, b_1 may be allowed to be zero. However, in this section we consider functions expanding into Stieltjes continued fractions, and $D_1(R) = s_0 \neq 0$ is one of the necessary conditions for such an expansion to exist by Corollary 1.35. At the same time, $s_0 \neq 0$ implies $b_1 = a_0 s_0 \neq 0$.

if $\deg q = \deg p$, that is, $b_0 \neq 0$, then

$$H(p,q) := \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.124}$$

The matrix $H(p,q)$ is called an *infinite matrix of Hurwitz type.*

Remark 1.41. The matrix $H(p,q)$ is of infinite rank since its submatrix obtained by deleting the even or odd rows of the original matrix is a triangular infinite matrix with $a_0 \neq 0$ on the main diagonal.

Together with the infinite matrix $H(p,q)$, we consider its specific finite submatrices:

Definition 1.42. Let the polynomials p and q be given by (1.2)–(1.3). If $\deg q < \deg p = n$, let $\mathcal{H}_{2n}(p,q)$ denote the following $2n \times 2n$-matrix:

$$\mathcal{H}_{2n}(p,q) = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n & 0 & 0 & \dots & 0 & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & \dots & 0 & 0 \\ 0 & b_1 & b_2 & \dots & b_{n-1} & b_n & 0 & \dots & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & a_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_n & 0 \\ 0 & 0 & 0 & \dots & 0 & b_1 & b_2 & \dots & b_n & 0 \\ 0 & 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}. \tag{1.125}$$

If $\deg q = \deg p = n$, let $\mathcal{H}_{2n+1}(p,q)$ denote the following $(2n+1) \times (2n+1)$-matrix

$$\mathcal{H}_{2n+1}(p,q) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & \dots & 0 & 0 \\ b_0 & b_1 & b_2 & \dots & b_{n-1} & b_n & 0 & \dots & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & a_n & \dots & 0 & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-2} & b_{n-1} & b_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_n & 0 \\ 0 & 0 & 0 & \dots & b_0 & b_1 & b_2 & \dots & b_n & 0 \\ 0 & 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}. \tag{1.126}$$

Both kinds of matrices $\mathcal{H}_{2n}(p,q)$ and $\mathcal{H}_{2n+1}(p,q)$ are called *finite matrices of Hurwitz type.* The leading principal minors of these matrices will be denoted by[9] $\Delta_j(p,q)$.

The infinite Hurwitz matrix $H(p,g)$ has an interesting factorization property:

Theorem 1.43. *If $g(z) = g_0 z^l + g_1 z^{l-1} + \cdots + g_l$, then*

$$H(p \cdot g, q \cdot g) = H(p,q)\mathcal{T}(g), \tag{1.127}$$

where $\mathcal{T}(g)$ is the infinite upper triangular Toeplitz matrix made of the coefficients of the polynomial g:

$$\mathcal{T}(g) = \begin{pmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & \dots \\ 0 & g_0 & g_1 & g_2 & g_3 & \dots \\ 0 & 0 & g_0 & g_1 & g_2 & \dots \\ 0 & 0 & 0 & g_0 & g_1 & \dots \\ 0 & 0 & 0 & 0 & g_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.128}$$

Here we set $g_i := 0$ for all $i > l$.

[9] That is, $\Delta_j(p,q)$ is the leading principal minor of the matrix $\mathcal{H}_{2n}(p,q)$ of order j if $\deg q < \deg p$. Otherwise (when $\deg q = \deg p$), $\Delta_j(p,q)$ denotes the leading principal minor of the matrix $\mathcal{H}_{2n+1}(p,q)$ of order j.

Proof. Straightforward multiplication of matrices $H(p,q)$ and $\mathcal{T}(g)$. $\square$

Denote by $\eta_j(p,q)$ the leading principal minor of the matrix $H(p,q)$ of order j $(j=1,2,\ldots)$. We now derive some key connections between these minors and the minors D_j, $\widehat{D}_j$ and ∇_{2j} we encountered before.

Lemma 1.44. *Let the polynomials p and q be defined by* (1.2)–(1.3) *and let*

$$R(z)=\frac{q(z)}{p(z)}=s_{-1}+\frac{s_0}{z}+\frac{s_1}{z^2}+\cdots$$

The following relations hold between the determinants $\eta_j(p,q)$ and $D_j(R),\widehat{D}_j(R),\nabla_{2j}(p,q)$ defined by (1.6), (1.9) *and* (1.12), *respectively.*

If $\deg q<\deg p$, then

$$\eta_{2j}(p,q)=\nabla_{2j}(p,q)=a_0^{2j}D_j(R),\qquad j=1,2,\ldots,\tag{1.129}$$

$$\eta_{2j+1}(p,q)=a_0\nabla_{2j}(zq,p)=(-1)^j a_0^{2j+1}\widehat{D}_j(R),\quad j=0,1,2,\ldots.\tag{1.130}$$

If $\deg q=\deg p$, then

$$\eta_{2j+1}(p,q)=b_0\nabla_{2j}(p,q)=b_0a_0^{2j}D_j(R),\qquad j=0,1,2,\ldots,\tag{1.131}$$

$$\eta_{2j}(p,q)=a_0b_0\nabla_{2j-2}(h,p)=(-1)^{j-1}b_0a_0^{2j-1}\widehat{D}_{j-1}(R),\quad j=1,2,\ldots,\tag{1.132}$$

where $h(z):=zq(z)-\dfrac{b_0}{a_0}zp(z)$ and $D_0(R):=\widehat{D}_0(R):=1$.

Proof. First, we prove the more complicated equalities (1.131)–(1.132). The formula (1.131) follows from (1.124), (1.12) and (1.13). To prove (1.132), we consider the function

$$G(z):=zR(z)-s_{-1}z=\frac{h(z)}{p(z)}=s_0+\frac{s_1}{z}+\frac{s_2}{z^2}+\cdots\tag{1.133}$$

Here we used the fact that $s_{-1}=\dfrac{b_0}{a_0}$. From (1.6) and (1.9) it follows that

$$D_j(G)=\widehat{D}_j(R),\qquad j=1,2,\ldots,\tag{1.134}$$

Next, for a fixed index $j=1,2,\ldots$, we have

$$\eta_{2j}(p,q)=\begin{vmatrix} b_0 & b_1 & b_2 & \dots & b_{j-1} & b_j & \dots & b_{2j-1}\\ 0 & a_0 & a_1 & \dots & a_{j-2} & a_{j-1} & \dots & a_{2j-2}\\ 0 & b_0 & b_1 & \dots & b_{j-2} & b_{j-1} & \dots & b_{2j-2}\\ 0 & 0 & a_0 & \dots & a_{j-3} & a_{j-2} & \dots & a_{2j-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_j\\ 0 & 0 & 0 & \dots & b_0 & b_1 & \dots & b_j\\ 0 & 0 & 0 & \dots & 0 & a_0 & \dots & a_{j-1}\end{vmatrix}.$$

For each $i=1,\ldots,j-1$, we now subtract the $(2i)$th row multiplied by $\dfrac{b_0}{a_0}$ from the $(2i{+}1)$st row to obtain

$$\eta_{2j}(p,q)=a_0b_0\nabla_{2j-2}(h,p)=a_0b_0(-1)^{j-1}\nabla_{2j-2}(p,h)\ .\tag{1.135}$$

This proves the first part of (1.132). Now from (1.13), (1.133) and (1.134) we have

$$\nabla_{2j}(p,h)=a_0^{2j}D_j(G)=a_0^{2j}\widehat{D}_j(R).$$

Combined with (1.135), this implies (1.132). The formulæ (1.129)–(1.130) can be proved analogously to (1.131)–(1.132), applying Theorem 1.5 to the functions $R(z)$ and $zR(z)$ and using Definition 1.40. $\square$

We next turn to finite matrices of Hurwitz type and consider some of their properties.

Theorem 1.45. *Let the polynomials p and q be defined as in* (1.2)–(1.3) *and let*

$$R(z) = \frac{q(z)}{p(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots$$

If $\deg q < \deg p$, *then*

$$\begin{aligned} \Delta_{2j-1}(p,q) &= a_0^{2j-1} D_j(R), \qquad && j = 1,2,\dots,n, && (1.136)\\ \Delta_{2j}(p,q) &= (-1)^j a_0^{2j} \widehat{D}_j(R), && j = 1,2,\dots,n. && (1.137)\end{aligned}$$

If $\deg q = \deg p$, *then*

$$\begin{aligned} \Delta_{2j}(p,q) &= a_0^{2j} D_j(R), \qquad && j = 1,2,\dots,n, && (1.138)\\ \Delta_{2j+1}(p,q) &= (-1)^j a_0^{2j+1} \widehat{D}_j(R), && j = 0,1,2,\dots,n, && (1.139)\end{aligned}$$

where $\widehat{D}_0(R) := 1$ *and the determinants* $D_j(R), \widehat{D}_j(R)$ *are defined by* (1.6), (1.9), *respectively.*

Proof. Apply Theorem 1.5 to the functions $R(z)$ and $zR(z)$ if $\deg q < \deg p$ or to the functions $R(z)$ and $zR(z) - z\dfrac{b_0}{a_0}$ if $\deg q = \deg p$, as in the proof of Lemma 1.44. ∎

We now summarize all previous results and add one more fact:

Theorem 1.46. *Given polynomials*

$$\begin{aligned} p(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad && a_1,\dots,a_n \in \mathbb{C}, \quad a_0 \neq 0,\\ q(z) &= b_0 z^n + b_1 z^{n-1} + \cdots + b_n, && b_0,\dots,b_n \in \mathbb{C},\end{aligned}$$

let the function

$$R(z) = \frac{q(z)}{p(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots$$

have exactly r poles ($r \leq n$), *counting multiplicities. The following conditions are equivalent:*

1) *the Hankel determinants $D_j(R)$ and $\widehat{D}_j(R)$ defined in* (1.6)–(1.9) *satisfy*

$$\begin{aligned} D_j(R) &\neq 0, \qquad && j = 1,2,\dots,r,\\ \widehat{D}_j(R) &\neq 0, && j = 1,2,\dots,r-1,\\ D_j(R) = \widehat{D}_j(R) &= 0, && j = r+1, r+2, \dots;\end{aligned}$$

   moreover, $\widehat{D}_r(R) = 0$ if and only if R has a pole at 0;

2) *the function R has a Stieltjes continued fraction expansion* (1.116) *whose coefficients c_j can be found by the formulæ* (1.114)–(1.115);

3) *the Euclidean algorithm applied to the polynomials $g_0(z) := p(z^2)$ and*[10] $g_1(z) := zq(z^2) - \dfrac{b_0}{a_0} zp(z^2)$ *is regular and* $\deg q \geq n-1$;

4) *the infinite matrix $H(p,q)$ factors as follows:*

   if $\deg q < \deg p$, *then*
$$H(p,q) = J(c_1)J(c_2)\cdots J(c_k)H(0,1)\mathcal{T}(g), \tag{1.140}$$

   if $\deg q = \deg p$, *then*
$$H(p,q) = J(c_0)J(c_1)\cdots J(c_k)H(0,1)\mathcal{T}(g), \tag{1.141}$$

[10] If $\deg q < \deg p$, then $b_0 = 0$, according to our convention.

where $k = 2r-1$, if $R(0)=\infty$, and $k=2r$, otherwise. The polynomial g is the general common divisor of p and q, the matrix $\mathcal{T}(g)$ is defined in (1.128) *and*

$$J(c) := \begin{pmatrix} c & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & c & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H(0,1) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.142}$$

The coefficients c_j are defined by the formulæ (1.114)–(1.115) *or, equivalently, as follows:*

if $\deg q < \deg p$, *then*

$$c_{2j-1} = \frac{\eta_{2j-1}^2(p,q)}{\eta_{2j-2}(p,q)\cdot\eta_{2j}(p,q)}, \qquad j=1,2,\dots,r, \tag{1.143}$$

$$c_{2j} = \frac{\eta_{2j}^2(p,q)}{\eta_{2j-1}(p,q)\cdot\eta_{2j}(p,q)}, \qquad j=1,2,\dots,\left\lfloor\frac{k}{2}\right\rfloor; \tag{1.144}$$

if $\deg q = \deg p$, *then*

$$c_{2j-1} = \frac{\eta_{2j}^2(p,q)}{\eta_{2j-1}(p,q)\cdot\eta_{2j+1}(p,q)}, \qquad j=1,2,\dots,r, \tag{1.145}$$

$$c_{2j} = \frac{\eta_{2j+1}^2(p,q)}{\eta_{2j}(p,q)\cdot\eta_{2j+2}(p,q)}, \qquad j=0,1,2,\dots,\left\lfloor\frac{k}{2}\right\rfloor. \tag{1.146}$$

Here we set $\eta_0(p,q):=1$ and $k:=2r-1$ if $R(0)=\infty$, whereas $k:=2r$ if $|R(0)|<\infty$.

Proof. Theorem 1.39 establishes the equivalence of conditions 1) and 2).

By Theorem 1.26, condition 3) is equivalent to the fact that the function $z\mapsto F(z)=\dfrac{g_1(z)}{g_0(z)}$ has a J-fraction expansion. At the same time, this is equivalent to condition 1), according to Theorem 1.26 and Lemma 1.34. Thus, we have proved the equivalence of conditions 1) and 3).

Now we prove that condition 4) follows from condition 3). As was shown above, whenever the Euclidean algorithm applied to the polynomials g_0 and g_1 is regular, the polynomials $f_0=p$ and $f_1=q$ satisfy (1.119)–(1.120) with the coefficients c_j determined by the formulæ (1.114)–(1.115). From Lemma 1.44 we see that the coefficients c_j can be found by the formulæ (1.143)–(1.144) or (1.145)–(1.146).

At first, let $\deg q<\deg p$ and let $g=\gcd(p,q)$. The algorithm (1.119)–(1.120) produces a sequence of polynomials $f_0,\ f_1,\ \dots,\ f_k$, where $k=2r$ if $p(0)\neq 0$ or $k=2r-1$ otherwise. Note that $f_0(z)=\dfrac{p(z)}{g(z)}$ and $f_1(z)=\dfrac{q(z)}{g(z)}$, that is, $\gcd(f_0,f_1)=1$. Consider any four consecutive polynomials $f_{j-1},\ f_j,\ f_{j+1},\ f_{j+2}$ ($j=1,\dots,2r-3$) in this sequence such that $\deg f_{j-1}>\deg f_j$, i.e., j is odd. Then the matrix $H(f_{j-1},f_j)$ (see Definition 1.40) satisfies

$$H(f_{j-1},f_j)=J(c_j)H(f_{j+1},f_j)=J(c_j)J(c_{j+1})H(f_{j+1},f_{j+2}). \tag{1.147}$$

This formula can be easily obtained by straightforward calculation using (1.119)–(1.121). For the polynomials f_{2r-2} and f_{2r-1}, the formula (1.147) has a different form:

if $k=2r-1$, then

$$H(f_{2r-2},f_{2r-1})=J(c_{2r-1})H(f_{2r},f_{2r-1})=J(c_{2r-1})H(0,1), \tag{1.148}$$

since $f_{2r-1}=f_k=\gcd(f_0,f_1)=1$, and therefore $f_{2r}(z)=f_{k+1}(z)\equiv 0$,

if $k = 2r$, then

$$\begin{aligned} H(f_{2r-2}, f_{2r-1}) &= J(c_{2r-1})H(f_{2r}, f_{2r-1}) = J(c_{2r-1})J(c_{2r})H(f_{2r+1}, f_{2r}) \\ &= J(c_{2r-1})J(c_{2r})H(0,1), \end{aligned} \tag{1.149}$$

since $f_{2r} = f_k = \gcd(f_0, f_1) = 1$, and $f_{2r+1}(z) = f_{k+1}(z) \equiv 0$.

Thus, from the formulæ (1.147)–(1.149) we obtain

$$H(f_0, f_1) = J(c_1)J(c_2)\cdots J(c_{k-1})J(c_k)H(0,1). \tag{1.150}$$

At the same time, Theorem 1.43 implies

$$H(p, q) = H(f_0, f_1)\mathcal{T}(g). \tag{1.151}$$

The formulæ (1.150)–(1.151) yield (1.140).

As for the case $\deg p = \deg q$, we denote $f_0(z) := \dfrac{p(z)}{g(z)}$, $f_1(z) := \dfrac{q(z) - c_0 p(z)}{g(z)}$, where $c_0 := \dfrac{b_0}{a_0}$, and find by straightforward calculation that

$$H(p, q) = J(c_0)H(f_0, f_1)\mathcal{T}(g). \tag{1.152}$$

Since $\deg f_0 > \deg f_1$, the matrix $H(f_0, f_1)$ satisfies (1.150). Thus, from (1.152) and (1.150) we obtain the factorization (1.141).

Conversely, if condition 4) holds, then we can reconstruct the algorithm (1.119)–(1.120) using the factorizations (1.140) or (1.141) as follows: the coefficients of the polynomials f_{j-1} and f_j (if $\deg f_{j-1} > \deg f_j$) are the entries in the first and the second rows, respectively, of the matrix

$$H(f_{j-1}, f_j) = J(c_j)J(c_{j+1})\cdots J(c_{k-1})J(c_k)H(0,1), \quad j = 1, \dots, k. \tag{1.153}$$

Here we have $f_0(z) = \dfrac{p(z)}{g(z)}$ and $f_1(z) = \dfrac{q(z) - c_0 p(z)}{g(z)}$, where[11] $c_0 = \dfrac{b_0}{a_0}$. Note that $g_0(z) = f_0(z^2)$ and $g_1(z) = z f_1(z^2)$ in our notation. As was shown above, the algorithm (1.119)–(1.120) for the polynomials f_0 and f_1 is equivalent to a regular Euclidean algorithm for the polynomials $g_0(z)$ and $g_1(z)$. Therefore, condition 4) implies condition 3). $\square$

Remark 1.47. Regarding equivalence classes of rational functions, we should note the following: Suppose that two rational functions R and G with exactly r poles each satisfy the inequalities (1.106)–(1.107). Then $R(z) \equiv G(z)$ if and only if

$$\begin{aligned} D_j(R) &= D_j(G), \\ \widehat{D}_j(R) &= \widehat{D}_j(G), \end{aligned} \qquad j = 1, 2, \dots, r,$$

since these equalities guarantee that the corresponding Stieltjes coefficients of R and G coincide by (1.114)–(1.115). We would like to remind the reader that, according to Theorem 1.20, the equality of the minors $D_j(R)$ and $D_j(G)$, for each j, *per se* does not guarantee that the functions R and G are equal.

2 Real rational functions and related topics

In this section we develop connections among several notions: the Euclidean algorithm and its variant, the Sturm algorithm (Section 2.1), Cauchy indices (Section 2.2), various representations of rational functions, and their associated Hankel minors. Those diverse topics turn out to be connected to the same basic question of counting roots or poles of rational functions. Throughout this section, we assume that all our rational functions are *real*.

[11] As was mentioned above, $b_0 = 0$ if $\deg p > \deg q$.

Thus, consider a *real* rational function

$$R(z) = \frac{q(z)}{p(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots, \quad s_i \in \mathbb{R}, \qquad i = -1, 0, 1, 2, \ldots. \tag{2.1}$$

where p and q are *real* polynomials

$$p(z) := a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_0, a_1, \ldots, a_n \in \mathbb{R},\ a_0 \neq 0, \tag{2.2}$$
$$q(z) := b_0 z^n + b_1 z^{n-1} + \cdots + b_n, \quad b_0, \ldots, b_n \in \mathbb{R}, \tag{2.3}$$

In what follows we assume that the function R has exactly $r = n - l$ poles (counting multiplicities) where l is the degree of the greatest common divisor of the polynomials p and q $(0 \leq l \leq n)$.

2.1 Sturm algorithm and Frobenius rule of signs for Hankel minors

For real polynomials $f_0 := p$ and $f_1 := q$, it is convenient to use a modification of the Euclidean algorithm, namely, its variant known as the Sturm algorithm. Suppose that we run the Euclidean algorithm (1.42) starting with the polynomials f_0 and f_1. If we denote

$$\begin{aligned} \widetilde{f}_j(z) &:= (-1)^{\frac{j(j-1)}{2}} f_j(z), \qquad j = 0, 1, \ldots, k, \\ \widetilde{q}_j(z) &:= (-1)^j q_j(z), \qquad j = 1, \ldots, k, \end{aligned}$$

then the polynomials $\widetilde{f}_j$ and $\widetilde{q}_k$ satisfy the following relations:

$$\widetilde{f}_{j-1}(z) = \widetilde{q}_j(z)\widetilde{f}_j(z) - \widetilde{f}_{j+1}(z), \qquad j = 0, 1, \ldots, k, \tag{2.4}$$

where $\widetilde{f}_{k+1}(z) \equiv 0$.

Definition 2.1. The relations (2.4) represent the so-called *Sturm algorithm.* If all the polynomials $\widetilde{q}_j$ are linear, the algorithm is called *regular.*

The polynomial f_k is the greatest common divisor of the polynomials f_0 and f_1. Thus, the Sturm algorithm also produces the greatest common divisor of two initial polynomials, but the Sturm form turns out to have advantages over the Euclidean form in the real case, as we will clarify later. Very roughly, in the real case signs of some quantities are more easily traced using the Sturm algorithm than the Euclidean algorithm. In the complex case, the issue of signs has no comparable significance.

In connection with the Sturm method, we mention briefly the so-called Sturm sequences, which, however, will not be used much in the sequel:

Definition 2.2. A sequence of polynomials g_0, g_1, ..., g_n is called a *Sturm sequence* on the interval (a, b) if

1) $g_0(z_*) = 0$ for some $z_* \in (a, b) \quad \Rightarrow \quad g_1(z_*) \neq 0$;

2) $g_j(z_*) = 0$ for some $z_* \in (a, b) \quad \Rightarrow \quad g_{j-1}(z_*) g_{j+1}(z_*) < 0 \quad$ for $j = 1, 2 \ldots, n-1$;

3) $g_n(z) \neq 0 \quad \forall z \in (a, b)$.

The sequence $\widetilde{f}_0$, $\widetilde{f}_1$, ..., $\widetilde{f}_r$ from the Sturm algorithm (2.4) is easily seen to be a Sturm sequence on any interval where $\widetilde{f}_r(z)$ does not vanish. Moreover, if we denote

$$\widetilde{h}_j(z) := \frac{\widetilde{f}_j(z)}{\widetilde{f}_r(z)}, \qquad j = 0, 1, \ldots, r,$$

then $\widetilde{h}_r(z) \equiv 1$ and the sequence $\widetilde{h}_0$, $\widetilde{h}_1$, ..., $\widetilde{h}_r$ is a Sturm sequence on the real axis.

Next, we must introduce several notions of sign changes for sequences of real numbers.

Definition 2.3. Given a sequence $\mathbf{t} := (t_0, t_1, \dots, t_n)$ without zeros, we say that its *number of sign changes* is the number of indices j between 1 and n satisfying $t_{j-1}t_j < 0$. The *number of sign retentions* is the number of indices j between 1 and n satisfying $t_{j-1}t_j > 0$.

For a sequence with zeros, the maximum number of sign changes obtainable by an appropriate choice of signs of any zero entry is called the *number of weak sign changes* and is denoted by $\mathrm{V}^+(\mathbf{t}) = \mathrm{V}^+(t_0, \dots, t_n)$. The minimum number so obtainable is called the *number of strong sign changes* and is denoted by $\mathrm{V}^-(\mathbf{t}) = \mathrm{V}^-(t_0, \dots, t_n)$. The *number of weak* $\mathrm{P}^+(\mathbf{t})$ *and strong* $\mathrm{P}^-(\mathbf{t})$ *sign retentions* can be defined correspondingly.

Note that the number of weak sign changes does not increase and the number of strong sign changes does not decrease under small perturbations of the elements of a given sequence. Also note that the number of strong sign changes can be determined simply by discarding all zero elements and counting the number of ordinary sign changes in the obtained sequence. Finally, note that the number of sign changes and the number of sign retentions (be it ordinary, weak, or strong, respectively) always sum up to n if $n+1$ is the length of the sequence:

$$\mathrm{V}^{\pm}(t_0, t_1, \dots, t_n) + \mathrm{P}^{\pm}(t_0, t_1, \dots, t_n) = n. \tag{2.5}$$

In the sequel, we will need only the notion of strong sign changes, so our discussion of weak sign changes above is included for completeness only. We will also need another important method of counting sign changes (and sign retentions) specifically introduced by Frobenius for sequences of Hankel minors[12] $(D_0(R), D_1(R), D_2(R), \dots, D_r(R))$ and $(\widehat{D}_0(R), \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_r(R))$ for a real rational function R. As usual, we set $D_0(R) := \widehat{D}_0(R) := 1$.

Rule 2.4 (Frobenius [29, 35])**.** *If, for some integers i and j $(0 \le i < j)$,*

$$D_i(R) \neq 0, \quad D_{i+1}(R) = D_{i+2}(R) = \cdots = D_{i+j}(R) = 0, \quad D_{i+j+1}(R) \neq 0, \tag{2.6}$$

then the number $\mathrm{V}^F(D_0(R), D_1(R), D_2(R), \dots, D_r(R))$ of Frobenius sign changes should be calculated by assigning signs as follows:

$$\operatorname{sign} D_{i+\nu}(R) = (-1)^{\frac{\nu(\nu-1)}{2}} \operatorname{sign} D_i(R), \quad \nu = 1, 2, \dots, j. \tag{2.7}$$

The number of Frobenius sign retentions $\mathrm{P}^F(D_0(R), D_1(R), D_2(R), \dots, D_r(R))$ is defined accordingly.

This assignment has an interesting property, which will be useful later.

Corollary 2.5. *If the condition* (2.6) *holds and if integers μ and ν $(i < \mu \le \nu \le j)$ have equal parities, then*

$$\mathrm{V}^F(D_\mu(R), D_{\mu+1}(R), \dots, D_\nu(R)) = \mathrm{P}^F(D_\mu(R), D_{\mu+1}(R), \dots, D_\nu(R)). \tag{2.8}$$

The Frobenius method of counting sign changes is so pervasive in the rest of this paper that we adopt the notational convention

$$\mathrm{V}(\mathbf{t}) := \mathrm{V}^F(\mathbf{t}), \qquad \mathrm{P}(\mathbf{t}) := \mathrm{P}^F(\mathbf{t}).$$

2.2 Cauchy indices and their properties

In this section, we introduce a special counter known as the Cauchy index. Consider a real rational function F, which may in principle have a pole at ∞, unlike the function R.

Definition 2.6. The quantity

$$\operatorname{Ind}_\omega(F) := \begin{cases} +1 & \text{if} \quad F(\omega - 0) < 0 < F(\omega + 0), \\ -1 & \text{if} \quad F(\omega - 0) > 0 > F(\omega + 0), \end{cases} \tag{2.9}$$

is called the *index* of the function F at its *real* pole ω of *odd* order.

[12]Since our function R is real, all its minors $D_j(R)$ and $\widehat{D}_j(R)$ and therefore the sequences $(D_0(R), D_1(R), D_2(R), \dots, D_r(R))$ and $(\widehat{D}_0(R), \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_r(R))$ are real.

We also set
$$\mathrm{Ind}_\omega(F) := 0 \tag{2.10}$$
if ω is a real pole of the function F of *even* order.

Suppose that the function F has m real poles in total, viz., $\omega_1 < \omega_2 < \cdots < \omega_m$.

Definition 2.7. The quantity
$$\mathrm{Ind}_a^b(F) := \sum_{i\,:\ a<\omega_i<b} \mathrm{Ind}_{\omega_i}(F). \tag{2.11}$$
is called the *Cauchy index* of the function F on the interval (a,b).

We are primarily interested in the quantity $\mathrm{Ind}_{-\infty}^{+\infty}(F)$, the Cauchy index of F on the real line. However, since the function F may have a pole at the point ∞, it is convenient for us to consider this pole as *real*. From this point of view, let us introduce the index at ∞ following, e.g., [11]. To do so, we consider the function F as a map on the *projective line* $\mathrm{P}\mathbb{R}^1 := \mathbb{R}^1 \cup \{\infty\}$ into itself. So, if the function F has a pole at ∞, then we let
$$\mathrm{Ind}_\infty(F) := \begin{cases} +1 & \text{if} \quad F(+\infty) < 0 < F(-\infty), \\ -1 & \text{if} \quad F(+\infty) > 0 > F(-\infty), \\ \ \ 0 & \text{if} \quad \operatorname{sign} F(+\infty) = \operatorname{sign} F(-\infty). \end{cases} \tag{2.12}$$
Thus, the generalized Cauchy index of the function F on the projective real line is
$$\mathrm{Ind}_{\mathrm{P}\mathbb{R}}(F) := \mathrm{Ind}_{-\infty}^{+\infty}(F) + \mathrm{Ind}_\infty(F). \tag{2.13}$$

Remark 2.8. Obviously, if the function F has no pole at ∞, then the generalized Cauchy index $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}(F)$ coincides with the usual Cauchy index $\mathrm{Ind}_{-\infty}^{+\infty}(F)$.

Following [11], we list a few properties of generalized Cauchy indices, which will be of use later.

First, note that a polynomial $q(z) = cz^\nu + \cdots$ $(\nu = \deg q)$ can be viewed as a rational function with a single pole at ∞, hence
$$\mathrm{Ind}_\infty(q) = \begin{cases} -\operatorname{sign} c & \text{if} \quad \nu \quad \text{is odd}, \\ \ \ 0 & \text{if} \quad \nu \quad \text{is even}. \end{cases} \tag{2.14}$$

The following theorem collects all properties of Cauchy indices that we will need.

Theorem 2.9 (see [11])**.** *Let F be a real rational function.*

1) If d is a real constant, then $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}(d+F) = \mathrm{Ind}_{\mathrm{P}\mathbb{R}}(F)$.

2) If q is a real polynomial and $|F(\infty)| < \infty$, then $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}(q+F) = \mathrm{Ind}_{-\infty}^{+\infty}(F) + \mathrm{Ind}_\infty(q)$.

3) If G and F are real rational functions that have no real poles in common, then
$$\mathrm{Ind}_{-\infty}^{+\infty}(F+G) = \mathrm{Ind}_{-\infty}^{+\infty}(F) + \mathrm{Ind}_{-\infty}^{+\infty}(G). \tag{2.15}$$

4) $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}\left(-\dfrac{1}{F}\right) = \mathrm{Ind}_{\mathrm{P}\mathbb{R}}(F)$.

Proof. Properties 1), 2) and 3) follow immediately from the definition of Cauchy indices (2.11)–(2.12) and from (2.14). The proof of Property 4) is reproduced from [11]. The projective line $\mathrm{P}\mathbb{R}^1$ is divided by two points, 0 and ∞, into its positive $(0,\infty)$ and negative $(-\infty,0)$ rays. Suppose that a variable z traverses $\mathrm{P}\mathbb{R}^1$ and returns to its starting point. Clearly, the number of crossings of $F(z)$ from $(-\infty,0)$ to $(0,\infty)$ must equal the number of reverse crossings. The crossings through ∞ occur at the poles of the function F; they are accounted for in the sum (2.13) with the appropriate sign. The crossings through 0 occur at the zeros of F, i.e., at the poles of the function $z \mapsto \dfrac{1}{F(z)}$, and they are accounted for in the analogous formula for $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}\left(\dfrac{1}{F}\right)$. As a result, $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}\left(\dfrac{1}{F}\right) + \mathrm{Ind}_{\mathrm{P}\mathbb{R}}(F) = 0$. ∎

Let us apply the Sturm algorithm (2.4) to the numerator and denominator of the fraction R (see (2.1)). As a result, we obtain another kind of continued fraction, which slightly differs from (1.46):

$$R(z) = s_{-1} + \cfrac{1}{q_1(z) - \cfrac{1}{q_2(z) - \cfrac{1}{q_3(z) - \cfrac{1}{\ddots - \cfrac{1}{q_k(z)}}}}} \tag{2.16}$$

Here the polynomials q_i have the form

$$q_j(z) = \alpha_j z^{n_j} + \cdots, \qquad \alpha_j \neq 0, \qquad j = 1, 2, \ldots, k, \tag{2.17}$$

where[13] $n_1 + n_2 + \cdots + n_k = r$ and $n_i \geq 1$, $i = 1, \ldots, k$. From Theorem 1.18 it is easy[14] to specialize the formula (1.62) to the function (2.16):

$$D_{n_1+n_2+\cdots+n_j}(R) = \prod_{i=1}^{j} (-1)^{\frac{n_i(n_i-1)}{2}} \cdot \prod_{i=1}^{j} \frac{1}{\alpha_i^{n_i+2\sum_{\rho=i+1}^{j} n_\rho}}, \qquad j = 1, 2, \ldots, k. \tag{2.18}$$

Applying property 1) and then inductively properties 4), 2) and 3) of Theorem 2.9 to the function (2.16), we get the following result:

Theorem 2.10 ([11])**.** *If a rational function R is represented by a continued fraction* (2.16)*, then*

$$\operatorname{Ind}_{\mathbb{PR}}(R) = -\sum_{j=1}^{k} \operatorname{Ind}_\infty(q_j). \tag{2.19}$$

This theorem implies an important fact recorded in Theorem 2.11 below. That fact is closely connected to the theory of quadratic forms. In fact, it was initially proved for rational functions with simple poles by Hermite using quadratic forms [45] (see also [44, pp.397–414]) and for arbitrary rational functions by Hurwitz [48] (see also [61, 36, 11]). Our proof differs from those proofs as well as from the proofs of Gantmacher in [36] and Barkovsky in [11] in that it does not use the theory of quadratic forms, only Frobenius Rule 2.4, some properties of continued fractions, and Theorem 2.10 about Cauchy indices.

Theorem 2.11 ([45, 44])**.** *If a rational function R with exactly r poles is represented by a series* (2.1)*, then*

$$\operatorname{Ind}_{-\infty}^{+\infty}(R) = r - 2\,\mathrm{V}(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)). \tag{2.20}$$

where the determinants $D_j(R)$ are defined in (1.6) *and where $D_0(R) := 1$.*

Proof. According to (2.5), the assertion of the theorem is equivalent to the following formula

$$\operatorname{Ind}_{-\infty}^{+\infty}(R) = \mathrm{P}(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)) - \mathrm{V}(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)). \tag{2.21}$$

Moreover, according to Remark 2.8, we have $\operatorname{Ind}_{\mathbb{PR}}(R) = \operatorname{Ind}_{-\infty}^{+\infty}(R)$.

First, let us assume that the function R has a J-fraction expansion obtained via a *regular* Sturm algorithm (see Definition 2.1), that is, where all the polynomials (2.17) are linear and $k = r$. Thus, $n_i = 1$ $(i = 1, 2, \ldots, r)$, and the formula (2.18) takes a simpler form (cf. (1.81))

$$D_j(R) = \prod_{i=1}^{j} \frac{1}{\alpha_i^{2j-2i+1}}, \qquad j = 1, 2, \ldots, r. \tag{2.22}$$

[13]Recall that r is the number of poles, counted with multiplicities, of the function R.

[14]One should replace G by $-G$ in the formula (1.49) and apply the formula (1.52) inductively to the function (2.16), taking into account that $(-1)^j D_j(G) = D_j(-G)$.

This implies the following formula:

$$\alpha_j = \frac{D_{j-1}(R)}{D_j(R)} \prod_{i=1}^{j-1} \frac{1}{\alpha_i^2}, \qquad j = 1, 2, \ldots, r. \tag{2.23}$$

In our (regular) case, from (2.14) and (2.19) we obtain

$$\operatorname{Ind}_{-\infty}^{+\infty}(R) = \sum_{j=1}^{r} \operatorname{sign} \alpha_j. \tag{2.24}$$

But (2.23) implies

$$\operatorname{sign} \alpha_j = \operatorname{sign} \frac{D_{j-1}(R)}{D_j(R)} = \mathrm{P}(D_{j-1}(R), D_j(R)) - \mathrm{V}(D_{j-1}(R), D_j(R)), \qquad j = 1, 2, \ldots, r. \tag{2.25}$$

Combining (2.24) and (2.25), we obtain (2.21).

Now let all polynomials q_j in (2.16) be of odd degrees, that is, let all the numbers n_j be *odd*. Then (2.14) and (2.19) imply

$$\operatorname{Ind}_{-\infty}^{+\infty}(R) = \sum_{j=1}^{k} \operatorname{sign} \alpha_j. \tag{2.26}$$

Using notation (1.68) from (2.18), we see that

$$\alpha_j^{n_j} = (-1)^{\frac{n_j(n_j-1)}{2}} \cdot \frac{D_{m_{j-1}}(R)}{D_{m_j}(R)} \cdot \prod_{i=1}^{j-1} \frac{1}{\alpha_i^{2n_j}}, \qquad j = 1, 2, \ldots, k. \tag{2.27}$$

Take into account that the numbers m_{j-1} and $m_j - 1$ have equal parities for every j, $1 \leq j \leq k$, since their difference $n_j - 1$ is even. Hence, according to Corollary 2.5 (see (2.8)), we have

$$\mathrm{P}(D_{m_{j-1}}(R), D_{m_{j-1}+1}(R), \ldots, D_{m_j-1}(R)) - \mathrm{V}(D_{m_{j-1}}(R), D_{m_{j-1}+1}(R), \ldots, D_{m_j-1}(R)) = 0. \tag{2.28}$$

The Frobenius Rule (2.7) gives

$$\operatorname{sign} D_{m_j-1}(R) = \operatorname{sign} D_{m_{j-1}+n_j-1}(R) = (-1)^{\frac{(n_j-1)(n_j-2)}{2}} \operatorname{sign} D_{m_j}(R), \qquad j = 1, 2, \ldots, k.$$

And from (2.27) we obtain

$$\begin{aligned}
\operatorname{sign} \alpha_j &= \operatorname{sign} \alpha_j^{n_j} = (-1)^{\frac{n_j(nj-1)}{2}} \operatorname{sign} \frac{D_{m_{j-1}}(R)}{D_{m_j}(R)} = \\
&= (-1)^{\frac{n_j(n_j-1)}{2}} \cdot (-1)^{\frac{(n_j-1)(n_j-2)}{2}} \cdot \operatorname{sign} \frac{D_{m_j-1}(R)}{D_{m_j}(R)} = \operatorname{sign} \frac{D_{m_j-1}(R)}{D_{m_j}(R)} = \\
&= \mathrm{P}(D_{m_j-1}(R), D_{m_j}(R)) - \mathrm{V}(D_{m_j-1}(R), D_{m_j}(R)), \qquad j = 1, 2, \ldots, k.
\end{aligned} \tag{2.29}$$

Here we used the condition that all n_j $(j = 1, 2, \ldots, k)$ are *odd*. From (2.28) and (2.29) we obtain

$$\begin{aligned}
\operatorname{sign} \alpha_j &= \mathrm{P}(D_{m_{j-1}}(R), D_{m_{j-1}+1}(R), \ldots, D_{m_j-1}(R), D_{m_j}(R)) - \\
&- \mathrm{V}(D_{m_{j-1}}(R), D_{m_{j-1}+1}(R), \ldots, D_{m_j-1}(R), D_{m_j}(R)), \qquad j = 1, 2, \ldots, k.
\end{aligned} \tag{2.30}$$

Substituting this formula into (2.26) yields (2.21).

Let us now consider the general case and let $1 \leq j_1 < j_2 < \cdots < j_\eta \leq k$ be the indices of those polynomials q_{i_1}, q_{i_2}, $\ldots$, q_{i_η} in (2.16) that have *odd* degrees. Then (2.14), (2.17) and (2.19) imply

$$\operatorname{Ind}_{-\infty}^{+\infty}(R) = \sum_{i=1}^{\eta} \operatorname{sign} \alpha_{j_i}. \tag{2.31}$$

Let i and j $(1 \le i < j \le k)$ be integers such that n_i and n_j are *odd* and $n_{i+1}, n_{i+2}, \dots, n_{j-1}$ are all *even*. Then the numbers $m_i, m_{i+1}, \dots, m_{j-1}, m_j - 1$ (see (1.68)) have equal parities and Corollary 2.5 yields

$$\mathrm{P}(D_{m_i}(R), D_{m_i+1}(R), \dots, D_{m_j-1}(R)) - \mathrm{V}(D_{m_i}(R), D_{m_i+1}(R), \dots, D_{m_j-1}(R)) = 0. \tag{2.32}$$

On the other hand, the formula (2.29) is valid in the present case, since n_j is *odd* by assumption. Therefore,

$$\begin{aligned} \operatorname{sign} \alpha_j &= \mathrm{P}(D_{m_i}(R), D_{m_i+1}(R), \dots, D_{m_j-1}(R), D_{m_j}(R)) - \\ &- \mathrm{V}(D_{m_i}(R), D_{m_i+1}(R), \dots, D_{m_j-1}(R), D_{m_j}(R)). \end{aligned} \tag{2.33}$$

Applied to the indices i_1, i_2, …, i_η , the formula (2.33) yields

$$\begin{aligned} \operatorname{sign} \alpha_{j_i} &= \mathrm{P}(D_{m_{j_{i-1}}}(R), D_{m_{j_{i-1}}+1}(R), \dots, D_{m_{j_i}}(R)) \\ &- \mathrm{V}(D_{m_{j_{i-1}}}(R), D_{m_{j_{i-1}}+1}(R), \dots, D_{m_{j_i}}(R)), \qquad i = 1, 2, \dots, \eta. \end{aligned} \tag{2.34}$$

If $j_\eta < k$, then the indices $n_{j_\eta+1}, n_{j_\eta+2}, \dots, n_k$ are all *even*, so all indices $m_{j_\eta}, m_{j_\eta+1}, m_{j_\eta+2}, \dots, m_k$ have equal parities, hence Corollary 2.5 implies

$$\mathrm{P}(D_{m_{j_\eta}}(R), D_{m_{j_\eta}+1}(R), \dots, D_{m_k}(R)) - \mathrm{V}(D_{m_{j_\eta}}(R), D_{m_{j_\eta}+1}(R), \dots, D_{m_k}(R)) = 0. \tag{2.35}$$

The formulæ (2.34)–(2.35) with (2.31) yield (2.21), as desired. □

Now we show how Theorem 2.11 can help in calculating the Cauchy index of a rational function if that function has a J-fraction or Stieltjes continued fraction expansion.

Theorem 2.12. *If a real rational function* (2.1) *with exactly r poles has a J-fraction expansion*

$$R(z) = s_{-1} + \cfrac{1}{\alpha_1 z + \beta_1 - \cfrac{1}{\alpha_2 z + \beta_2 - \cfrac{1}{\alpha_3 z + \beta_3 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_r z + \beta_r}}}}}$$

and m $(0 \le m \le r)$ is the number of negative coefficients α_j $(j = 1, 2, \dots, r)$, then

$$\operatorname{Ind}_{-\infty}^{+\infty}(R) = r - 2m. \tag{2.36}$$

Proof. From (2.25) we get that α_j is negative if and only if there is a sign change in the sequence $\{D_{j-1}(R), D_j(R)\}$. Consequently, $m = \mathrm{V}(D_0(R), D_1(R), D_2(R), \dots, D_r(R))$, and (2.36) follows from (2.20). Also (2.36) follows from (2.24). □

Theorem 2.13. *If a real rational function* (2.1) *with exactly r poles has a Stieltjes continued fraction expansion* (1.116) *and if m $(0 \le m \le r)$ is a number of negative coefficients c_{2j-1} $(j = 1, 2, \dots, r)$, then the Cauchy index $\operatorname{Ind}_{-\infty}^{+\infty}(R)$ can be found by the formula* (2.36).

Proof. The formula (2.36) follows from (1.115) and (2.20). □

Using Theorem 2.11, we can also produce formulæ for calculating the Cauchy index of a rational function on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

Theorem 2.14. *Let a real rational function R with exactly r poles have a series expansion* (2.1). *Then the indices of the function R on the intervals $(0, +\infty)$ and $(-\infty, 0)$ can be found from the following formulæ:*

- *if $|R(0)| < \infty$, then*

$$\operatorname{Ind}_{0}^{+\infty}(R) = r - \left[\mathrm{V}(1, D_1(R), D_2(R), \dots, D_r(R)) + \mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_r(R))\right], \tag{2.37}$$

$$\operatorname{Ind}_{-\infty}^{0}(R) = \mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_r(R)) - \mathrm{V}(1, D_1(R), D_2(R), \dots, D_r(R)), \tag{2.38}$$

- *if $z=0$ is a pole of R of order ν, then*

$$\operatorname{Ind}_{0}^{+\infty}(R) \;=\; r-\frac{1+\sigma_1}{2}-\Big[\mathrm{V}(1,D_1(R),D_2(R),\dots,D_r(R))+\mathrm{V}(1,\widehat{D}_1(R),\widehat{D}_2(R),\dots,\widehat{D}_{r-1}(R))\Big], \quad (2.39)$$

$$\operatorname{Ind}_{-\infty}^{0}(R) \;=\; \frac{1-\sigma_2}{2}+\mathrm{V}(1,\widehat{D}_1(R),\widehat{D}_2(R),\dots,\widehat{D}_{r-1}(R))-\mathrm{V}(1,D_1(R),D_2(R),\dots,D_r(R)), \quad (2.40)$$

where $\sigma_1=\operatorname{sign}\left(\lim\limits_{z\to0}z^{\nu}R(z)\right)$, *and* $\sigma_2=\begin{cases}\sigma_1 & \text{if } \operatorname{Ind}_0 R(z)\neq 0,\\ -\sigma_1 & \text{if } \operatorname{Ind}_0 R(z)=0.\end{cases}$

Proof. At first, let the function R have no pole at 0, that is, suppose that $|R(0)|<\infty$. Represent $R(z)$ in the form

$$R(z)=R^{(1)}(z)+R^{(2)}(z)+G(z), \quad (2.41)$$

where the function G has no real poles of odd order, that is, where $\operatorname{Ind}_{-\infty}^{+\infty}(G)=0$, and

$$\begin{aligned}R^{(1)}(z) &= \sum_{j=1}^{m_-}\left(\frac{A_{l_j}^{(j)}}{(z+\omega_j)^{l_j}}+\frac{A_{l_j-1}^{(j)}}{(z+\omega_j)^{l_j-1}}+\cdots+\frac{A_1^{(j)}}{z+\omega_j}\right),\\ R^{(2)}(z) &= \sum_{j=m_-+1}^{m_-+m_+}\left(\frac{A_{l_j}^{(j)}}{(z-\omega_j)^{l_j}}+\frac{A_{l_j-1}^{(j)}}{(z-\omega_j)^{l_j-1}}+\cdots+\frac{A_1^{(j)}}{z-\omega_j}\right),\end{aligned} \quad (2.42)$$

where all l_j are odd and $\omega_j>0$. Here m_- (m_+) is the numbers of negative (positive) poles of odd order of the function R. It follows from Definitions 2.6 and 2.7 and from the formulæ (2.41)–(2.42) that

$$\begin{aligned}\operatorname{Ind}_{-\infty}^{0}(R) &= \operatorname{Ind}_{-\infty}^{0}(R^{(1)}) = \sum_{j=1}^{m_-}\operatorname{sign}\left(A_{l_j}^{(j)}\right),\\ \operatorname{Ind}_{0}^{+\infty}(R) &= \operatorname{Ind}_{0}^{+\infty}(R^{(2)}) = \sum_{j=m_-+1}^{m_-+m_+}\operatorname{sign}\left(A_{l_j}^{(j)}\right).\end{aligned} \quad (2.43)$$

Now consider the function $z\mapsto F(z)=zR(z)$. From (2.1) and (2.41) we have

$$F(z)=zR^{(1)}(z)+zR^{(2)}(z)+zG(z)=s_{-1}z+s_0+\frac{s_1}{z}+\frac{s_2}{z^2}+\frac{s_3}{z^3}+\cdots, \quad (2.44)$$

where

$$\begin{aligned}zR^{(1)}(z) &= -\sum_{j=1}^{m_-}\left(\frac{\omega_jA_{l_j}^{(j)}}{(z+\omega_j)^{l_j}}+\frac{\omega_jA_{l_j-1}^{(j)}-A_{l_j}^{(j)}}{(z+\omega_j)^{l_j-1}}+\cdots+\frac{\omega_jA_1^{(j)}-A_2^{(j)}}{z+\omega_j}\right)+\sum_{j=1}^{m_-}A_1^{(j)},\\ zR^{(2)}(z) &= \sum_{j=m_-+1}^{m_-+m_+}\left(\frac{\omega_jA_{l_j}^{(j)}}{(z-\omega_j)^{l_j}}+\frac{\omega_jA_{l_j-1}^{(j)}-A_{l_j}^{(j)}}{(z-\omega_j)^{l_j-1}}+\cdots+\frac{\omega_jA_1^{(j)}-A_2^{(j)}}{z-\omega_j}\right)+\sum_{j=m_-+1}^{m_-+m_+}A_1^{(j)},\end{aligned} \quad (2.45)$$

From Definitions 2.6 and 2.7 and from the formulæ (2.43)–(2.45) we obtain

$$\operatorname{Ind}_{-\infty}^{0}(F) \;=\; \operatorname{Ind}_{-\infty}^{0}(zR^{(1)}) \;=\; -\sum_{j=1}^{m_-}\operatorname{sign}\left(\omega_jA_{l_j}^{(j)}\right) \;=\; -\sum_{j=1}^{m_-}\operatorname{sign}\left(A_{l_j}^{(j)}\right) \;=\; -\operatorname{Ind}_{-\infty}^{0}(R), \quad (2.46)$$

$$\operatorname{Ind}_{0}^{+\infty}(F) \;=\; \operatorname{Ind}_{0}^{+\infty}(zR^{(2)}) \;=\; \sum_{j=m_-+1}^{m_-+m_+}\operatorname{sign}\left(\omega_jA_{l_j}^{(j)}\right) \;=\; \sum_{j=m_-+1}^{m_-+m_+}\operatorname{sign}\left(A_{l_j}^{(j)}\right) \;=\; \operatorname{Ind}_{0}^{+\infty}(R), \quad (2.47)$$

since all ω_j are positive.

Theorems 2.9 and 2.11 and the formulaæ (2.46)–(2.47) yield

$$\begin{aligned}
\operatorname{Ind}_{-\infty}^{+\infty}(R) &= \operatorname{Ind}_{-\infty}^{0}(R) + \operatorname{Ind}_{0}^{+\infty}(R) = r - 2\,\mathrm{V}(1, D_1(R), D_2(R), \dots, D_r(R)),\\
\operatorname{Ind}_{-\infty}^{+\infty}(F) &= \operatorname{Ind}_{-\infty}^{0}(F) + \operatorname{Ind}_{0}^{+\infty}(F) = -\operatorname{Ind}_{-\infty}^{0}(R) + \operatorname{Ind}_{0}^{+\infty}(R) \ = \\
&= r - 2\,\mathrm{V}(1, D_1(F), D_2(F), \dots, D_r(F)) = r - 2\,\mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_r(R)),
\end{aligned}$$

These formulæ imply (2.37)–(2.38).

If the function R has a pole of order ν (≥ 1) at 0, that is, if $R(0) = \infty$, then instead of (2.41) and (2.44) we obtain

$$\begin{aligned}
R(z) &= R^{(1)}(z) + R^{(2)}(z) + \frac{C}{z^{\nu}} + G(z),\\
F(z) &= zR^{(1)}(z) + zR^{(2)}(z) + \frac{C}{z^{\nu-1}} + zG(z),
\end{aligned}$$

where $C = \lim\limits_{z\to 0} z^{\nu} R(z)$, the functions $R^{(1)}$ and $R^{(2)}$ are the same as in (2.42), and $\operatorname{Ind}_{-\infty}^{+\infty}(G) = 0$. The formulæ (2.46)–(2.47) remain the same.

If ν is odd, then $\operatorname{Ind}_0(R) = \sigma_1 \neq 0$ but $\operatorname{Ind}_0(F) = 0$, $D_r(F) = \widehat{D}_r(R) = 0$ (see Corollary 1.4), and

$$\begin{aligned}
\operatorname{Ind}_{-\infty}^{+\infty}(R) &= \operatorname{Ind}_{-\infty}^{0}(R) + \operatorname{Ind}_{0}^{+\infty}(R) + \sigma_1 = r - 2\,\mathrm{V}(1, D_1(R), D_2(R), \dots, D_r(R)),\\
\operatorname{Ind}_{-\infty}^{+\infty}(F) &= \operatorname{Ind}_{-\infty}^{0}(F) + \operatorname{Ind}_{0}^{+\infty}(F) = -\operatorname{Ind}_{-\infty}^{0}(R) + \operatorname{Ind}_{0}^{+\infty}(R) \ = \\
&= r - 1 - 2\,\mathrm{V}(1, D_1(F), D_2(F), \dots, D_{r-1}(F)) \ = \\
&= r - 1 - 2\,\mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_{r-1}(R)).
\end{aligned} \tag{2.48}$$

If ν is even, then $\operatorname{Ind}_0(R) = 0$ but $\operatorname{Ind}_0(F) = \sigma_1 \neq 0$, $D_r(F) = \widehat{D}_r(R) = 0$, and

$$\begin{aligned}
\operatorname{Ind}_{-\infty}^{+\infty}(R) &= \operatorname{Ind}_{-\infty}^{0}(R) + \operatorname{Ind}_{0}^{+\infty}(R) = r - 2\,\mathrm{V}(1, D_1(R), D_2(R), \dots, D_r(R)),\\
\operatorname{Ind}_{-\infty}^{+\infty}(F) &= \operatorname{Ind}_{-\infty}^{0}(F) + \operatorname{Ind}_{0}^{+\infty}(F) + \sigma_1 = -\operatorname{Ind}_{-\infty}^{0}(R) + \operatorname{Ind}_{0}^{+\infty}(R) \ = \\
&= r - 1 - 2\,\mathrm{V}(1, D_1(F), D_2(F), \dots, D_{r-1}(F)) \ = \\
&= r - 1 - 2\,\mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \dots, \widehat{D}_{r-1}(R)).
\end{aligned} \tag{2.49}$$

The formulæ (2.48)–(2.49) give (2.39)–(2.40). ∎

If a rational function R has a Stieltjes fraction expansion, then using Theorems 2.14 and formulæ (1.114), we can establish relations between $\operatorname{Ind}_{-\infty}^{0}(R)$, $\operatorname{Ind}_{0}^{+\infty}(R)$ and the signs of the coefficients in the Stieltjes fraction of R.

Theorem 2.15. *Suppose that a real rational function R with exactly r poles has a Stieltjes continued fraction expansion* (1.116). *Then the indices of the function R on the intervals* $(0, +\infty)$ *and* $(-\infty, 0)$ *can be found by the following formulæ:*

if $|R(0)| < \infty$, then

$$\begin{aligned}
\operatorname{Ind}_{0}^{+\infty}(R) &= n_e - n_o, && (2.50)\\
\operatorname{Ind}_{-\infty}^{0}(R) &= r - [n_o + n_e], && (2.51)
\end{aligned}$$

where n_o is the number of negative coefficients c_{2i-1}, $i = 1, 2, \dots, r$, and n_e is the number of negative coefficients c_{2i}, $i = 1, 2, \dots, r$.

if $z = 0$ is a pole of R of order ν, then

$$\begin{aligned}
\operatorname{Ind}_{0}^{+\infty}(R) &= \frac{1-\delta_1}{2} + n_e - n_o, && (2.52)\\
\operatorname{Ind}_{-\infty}^{0}(R) &= r - \frac{1+\delta_2}{2} - [n_o + n_e], && (2.53)
\end{aligned}$$

where n_o is the number of negative coefficients c_{2i-1}, $i = 1, 2, \ldots, r$, n_e is the number of negative coefficients c_{2i}, $i = 1, 2, \ldots, r-1$, $\delta_1 = \operatorname{sign}\left(\lim\limits_{z\to 0} z^{\nu} R(z)\right)$, and $\delta_2 = \begin{cases} \delta_1 & \text{if} \quad \operatorname{Ind}_0(R) \neq 0, \\ -\delta_1 & \text{if} \quad \operatorname{Ind}_0(R) = 0. \end{cases}$

Proof. From the formulæ (1.114)–(1.115) we obtain

$$n_{\mathrm{o}} = \mathrm{V}(1, D_1(R), D_2(R), \ldots, D_r(R)),$$

$$n_{\mathrm{e}} = \mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \ldots, \widehat{D}_k(R)),$$

where $k = r$ if $|R(0)| < \infty$, and $k = r - 1$ if $R(0) = \infty$. The assertion of the theorem follows from these formulæ and from Theorem 2.14. ∎

3 Rational functions mapping the upper half-plane to the lower half-plane

3.1 General theory

Now we specialize general properties of complex and real rational functions stated in the previous sections to the following very important class of functions:

Definition 3.1. A rational function R is called an *R-function of negative type* (respectively, *positive type*) if it maps the upper half-plane of the complex plane to the lower half-plane (respectively, to itself):

$$\operatorname{Im} z > 0 \Longrightarrow \operatorname{Im} R(z) < 0 \quad \text{—} \quad \textit{negative type};$$

$$\operatorname{Im} z > 0 \Longrightarrow \operatorname{Im} R(z) > 0 \quad \text{—} \quad \textit{positive type}.$$

The name *R-function* appears first in the works of M.G. Krein and his progeny in connection with the theory of Stieltjes string and Stieltjes continued fractions (see, for example, [34, 52]). Below we discuss several well-known and some new relationships between R-functions and continued fractions of Stieltjes type (see Definition 1.37). By now, these functions, as well as their meromorphic analogues, have been considered by many authors and have acquired various names. For instance, these functions are called *strongly real functions* in the monograph [90] due to their property to take real values *only* for real values of the argument (more general and detailed consideration can be found in [19], see also Theorem 3.4 below).

Remark 3.2. In the sequel, we will deal with R-functions of negative type only. But if an R-function F is of negative type, then the function $-F$ is evidently an R-function of positive type. Hence all results obtained for R-functions of negative type can be easy reformulated for R-functions of positive type.

At first, let us prove one necessary condition for a rational function to be an R-function.

Lemma 3.3. *If a real rational function*

$$z \mapsto R(z) = \frac{q(z)}{p(z)}$$

is an R-function of negative type, where p and q are real polynomials, then

$$|\deg p - \deg q| \leq 1, \tag{3.1}$$

and

$$R(z) = -\alpha z + \beta + \widetilde{R}(z), \qquad \alpha \geq 0, \;\; \beta \in \mathbb{R}, \;\; \widetilde{R}(\infty) = 0. \tag{3.2}$$

If $\alpha = 0$, then $\widetilde{R}$ is an R-function of negative type.

Proof. Indeed,

$$R(z) = h(z) + \widetilde{R}(z), \qquad \widetilde{R}(\infty) = 0,$$

where $h(z) = cz^j + \cdots$ is a real polynomial of some degree j. Obviously, for $z \in \mathbb{C}$ such that $\operatorname{Im} z > 0$ and sufficiently large, we have

$$\operatorname{sign}(\operatorname{Im} R(z)) = \operatorname{sign}(\operatorname{Im} h(z)) = \operatorname{sign}(c \sin(j\varphi)),$$

where $\varphi = \arg z$. If $j \geq 2$, then $\operatorname{Im} R(z)$ takes both positive and negative values for some z from the upper complex half-plane, say, for $\arg z = \dfrac{\pi}{2j}$ and $\arg z = \dfrac{3\pi}{2j}$. Therefore, the degree of h is necessarily at most 1: $h(z) = -\alpha z + \beta$, where $\alpha, \beta \in \mathbb{R}$. But $\operatorname{Im} h(z) = -\alpha \operatorname{Im} z$, thus, α must be nonnegative. Moreover, if $\alpha = 0$, then $\operatorname{Im} R(z) = \operatorname{Im} \widetilde{R}(z) < 0$ for z such that $\operatorname{Im} z > 0$, and $\widetilde{R}(z)$ is an R-function of negative type.

Thus, if $\deg q > \deg p$ and the function $R = \dfrac{q}{p}$ is an R-function of negative type, then

$$\deg q \leq \deg p + 1. \tag{3.3}$$

Let $\deg q < \deg p$ and let the function $R = \dfrac{q}{p}$ be an R-function of negative type. Then the function $-\dfrac{p}{q}$ is an R-function of negative type too and, therefore,

$$\deg q \leq \deg p + 1. \tag{3.4}$$

From (3.3)–(3.4) we obtain (3.1). □

The following theorem lists the most important properties of R-functions. Parts of this theorem can be found in [75, 61, 19, 36, 8, 7, 11, 90].

Theorem 3.4. *Let p and q be real and coprime*[15] *polynomials satisfying* (3.1). *For the real rational function*

$$z \mapsto R(z) = \frac{q(z)}{p(z)}$$

with exactly $n = \deg p$ poles, the following conditions are equivalent:

1) *R is an R-function of negative type:*

$$\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} R(z) < 0; \tag{3.5}$$

2) *The function R can be represented in the form*

$$R(z) = -\alpha z + \beta + \sum_{j=1}^{n} \frac{\gamma_j}{z - \omega_j}, \qquad \alpha \geq 0,\ \beta \in \mathbb{R},\ \omega_j \in \mathbb{R}, \quad j = 1, 2, \ldots, n, \tag{3.6}$$

where

$$\gamma_j = \frac{q(\omega_j)}{p'(\omega_j)} > 0, \quad j = 1, \ldots, n; \tag{3.7}$$

3) *The index of the function R is maximal:*

$$\operatorname{Ind}_{\mathbb{PR}}(R) = \max(\deg p, \deg q)\,; \tag{3.8}$$

4) *The function R has a J-fraction expansion:*

$$R(z) = -\alpha z + \beta + \cfrac{1}{\alpha_1 z + \beta_1 - \cfrac{1}{\alpha_2 z + \beta_2 - \cfrac{1}{\alpha_3 z + \beta_3 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_n z + \beta_n}}}}}, \tag{3.9}$$

where $\alpha_j > 0, \beta_j \in \mathbb{R}$, $\alpha \geq 0$ and $\beta \in \mathbb{R}$;

[15] This condition is introduced for simplicity and just means that the number of poles of the function R equals the number of zeros of the polynomial p.

5) *The polynomials p and q have only real roots and satisfy the inequality*

$$p(\omega)q'(\omega) - p'(\omega)q(\omega) < 0 \quad \textit{for all} \quad \omega \in \mathbb{R}; \tag{3.10}$$

6) *The roots of the polynomials p and q are real, simple and interlacing, that is, between any two consecutive roots of one of the polynomials there is exactly one root of the other polynomial, and*

$$\exists\, \omega \in \mathbb{R}: \quad p(\omega)q'(\omega) - p'(\omega)q(\omega) < 0; \tag{3.11}$$

7) *The polynomial*

$$z \mapsto g(z) = \lambda p(z) + \mu q(z), \tag{3.12}$$

has only real zeros for any real λ and μ, $\lambda^2 + \mu^2 \neq 0$, and the condition (3.11) *is satisfied;*

8) *The function $R(z)$ has real values only for real z:*

$$R(z) \in \mathbb{R} \quad \Longrightarrow \quad z \in \mathbb{R}, \tag{3.13}$$

and

$$\exists\, \omega \in \mathbb{R}: \quad -\infty < R'(\omega) < 0; \tag{3.14}$$

9) *Let the function R be represented by the series*

$$R(z) = -\alpha z + \beta + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots \tag{3.15}$$

with $\alpha \geq 0$ and $\beta \in \mathbb{R}$. The following inequalities hold

$$D_j(R) > 0, \quad j = 1, 2, \ldots, n, \tag{3.16}$$

where the determinants $D_j(R)$ are defined in (1.6).

The equivalence of conditions 1) and 2) is usually called the Chebotarev theorem [97, 19]. The equivalence of conditions 1) and 9) in case of meromorphic functions is the famous Grommer theorem [39, 4, 19]. The equivalence between 2) and 4) was also proved by Grommer [39] (see also [101, 65]). Finally, the equivalence of 1) and 6) is a modification of the famous Hermite-Biehler theorem (for example, [61, 36]).

Proof. The scheme of our proof is as follows:

$$\begin{array}{ccccccccc}
1) & \Longrightarrow & 2) & \Longrightarrow & 3) & \Longrightarrow & 4) & \Longleftrightarrow & 9) \\
\Uparrow & & & & & & \Downarrow & & \\
8) & \Longleftarrow & 7) & \Longleftarrow & 6) & \Longleftarrow & 5) & &
\end{array}$$

1) $\Longrightarrow$ 2) Let the function R satisfy (3.5). Lemma 3.3 guarantees that R can be represented in the form (3.2). Thus, we have to prove that the function $\widetilde{R}$ in the representation (3.2) has the following form

$$\widetilde{R}(z) = \frac{\tilde{q}(z)}{p(z)} = \sum_{j=1}^{n} \frac{\gamma_j}{z - \omega_j}, \qquad \omega_j \in \mathbb{R}, \quad j = 1, 2, \ldots, n, \tag{3.17}$$

where[16] $\tilde{q}(z) = q(z) - (\alpha z + \beta)p(z)$ and

$$\gamma_j = \frac{\tilde{q}(\omega_j)}{p'(\omega_j)} = \frac{q(\omega_j)}{p'(\omega_j)} > 0, \qquad j = 1, \ldots, n. \tag{3.18}$$

At first, assume that the function R and, therefore, the function $\widetilde{R}$ have a nonreal pole λ of some multiplicity $j (\geq 1)$. Then for $z = \lambda + \varepsilon$, $|\varepsilon| \to 0$, we have

$$\operatorname{sign}\left(\operatorname{Im} R(z)\right) = \operatorname{sign}\left(\operatorname{Im} \widetilde{R}(z)\right) = \operatorname{sign}\left(\operatorname{Im} \frac{c}{\varepsilon^j}\right) = \operatorname{sign}\left(\sin(\arg c - j \cdot \arg \varepsilon)\right).$$

[16]Here α and β are the numbers from the representation (3.2).

From these equalities one can see that we can obviously choose such complex numbers ε_1 and ε_1 that $z_1 = \lambda + \varepsilon_1$ and $z_2 = \lambda + \varepsilon_2$ are from the upper half plane, but $\operatorname{sign}(\operatorname{Im} R(z_1)) = -\operatorname{sign}(\operatorname{Im} R(z_2))$. It contradicts with (3.5). Thus, the functions R and $\widetilde{R}$ have only real poles.

Now let us assume that R and $\widetilde{R}$ have a real pole μ of some multiplicity $j \geq 2$. And let $z = \mu + \varepsilon$. Since μ is a real number, we have $\operatorname{Im} z = \operatorname{Im} \varepsilon$. If we take z from the upper half-plane of complex plane and sufficiently close to μ, that is, $0 < \arg\varepsilon < \pi$ and $|\varepsilon| \to 0$, then

$$\operatorname{sign}(\operatorname{Im} R(z)) = \operatorname{sign}\left(\operatorname{Im} \widetilde{R}(z)\right) = \operatorname{sign}\left(\operatorname{Im} \frac{A}{\varepsilon^j}\right) = -\operatorname{sign}(A\sin(j\cdot\arg\varepsilon)).$$

These equalities show that if we choose ε_1 and ε_2 such that $\arg\varepsilon_1 = \dfrac{\pi}{2j}$ and $\arg\varepsilon_2 = \dfrac{3\pi}{2j}$, then $z_1 = \mu + \varepsilon_1$ and $z_2 = \mu + \varepsilon_2$ both belong to the upper half-plane, if $j \geq 0$, but $\operatorname{sign}(\operatorname{Im} R(z_1)) = -\operatorname{sign}(\operatorname{Im} R(z_2))$. Thus, the function R (and the function $\widetilde{R}$) has only simple real poles, and $\widetilde{R}$ satisfies (3.17), where the residues γ_j are determined by the formula

$$\gamma_j = \lim_{z\to\omega_j} (R(z)(z-a)) = \lim_{z\to\omega_j} \frac{\tilde{q}(z)(z-a)}{p(z)} = \frac{\tilde{q}(\omega_j)}{p'(\omega_j)} = \frac{q(\omega_j)}{p'(\omega_j)}, \qquad j = 1, \ldots, n.$$

For z sufficiently close to a pole ω_j $(j = 1, 2, \ldots, n)$, we have

$$\operatorname{sign}(\operatorname{Im} R(z)) = \operatorname{sign}\left(\operatorname{Im} \widetilde{R}(z)\right) = \operatorname{sign}\left(\operatorname{Im} \frac{\gamma_j}{z-\omega_j}\right) = -\operatorname{sign}(\gamma_j \operatorname{Im} z).$$

These equalities combined with (3.5) yield the positivity of all γ_j. Consequently, the function $\widetilde{R}$ satisfies (3.17)–(3.18), but R has the representation (3.6)–(3.7).

2) $\Longrightarrow$ 3) Let the function R satisfy (3.6). If $\deg p \geq \deg q$, then $\alpha = 0$ and $\max(\deg p, \deg q) = \deg p = n$. In this case, R has no pole at ∞, so from (2.9) we obtain

$$\operatorname{Ind}_{\mathrm{P}\mathbb{R}}(R) = \operatorname{Ind}_{-\infty}^{+\infty}(R) = n = \max(\deg p, \deg q).$$

If $\deg q = \deg p + 1$, that is, $\alpha > 0$, then R has a pole at ∞, so according to (2.9), and (2.12)–(2.14),

$$\operatorname{Ind}_{\mathrm{P}\mathbb{R}}(R) = \operatorname{sign}\alpha + \operatorname{Ind}_{-\infty}^{+\infty}(R) = 1 + n = \deg q = \max(\deg p, \deg q).$$

3) $\Longrightarrow$ 4) Since the function R satisfies (3.1), it can be represented as follows:

$$R(z) = -\alpha z + \beta + \widetilde{R}(z), \tag{3.19}$$

where $\alpha, \beta \in \mathbb{R}$ and $\widetilde{R}(\infty) = 0$. Then from (2.12)–(2.15) we have

$$\operatorname{Ind}_{\mathrm{P}\mathbb{R}}(R) = \operatorname{sign}\alpha + \operatorname{Ind}_{-\infty}^{+\infty}(R) = \operatorname{sign}\alpha + \operatorname{Ind}_{-\infty}^{+\infty}(\widetilde{R}), \tag{3.20}$$

therefore, according to (2.9) and (2.12),

if $\deg q = \deg p + 1$, then

$$-n - 1 \leq \operatorname{Ind}_{\mathrm{P}\mathbb{R}}(R) \leq n + 1, \tag{3.21}$$

if $\deg q \leq \deg p$, then

$$-n \leq \operatorname{Ind}_{\mathrm{P}\mathbb{R}}(R) \leq n, \tag{3.22}$$

The condition (3.8) is equivalent to

$$\operatorname{Ind}_{\mathrm{P}\mathbb{R}}(R) = \begin{cases} n+1 & \text{if} \quad \deg q = \deg p + 1; \\ n & \text{if} \quad \deg q \leq \deg p. \end{cases} \tag{3.23}$$

From (3.20)–(3.23) we obtain that $\alpha \geq 0$ in (3.19) and $\operatorname{Ind}_{-\infty}^{+\infty}(R) = \operatorname{Ind}_{-\infty}^{+\infty}(\widetilde{R}) = n$.

Let us expand the function R into a continued fraction (2.16). Then Theorem 2.10 yields

$$\mathrm{Ind}_{\mathrm{P}\mathbb{R}}(\widetilde{R}) = -\sum_{j=1}^{k} \mathrm{Ind}_{\infty}(q_j),$$

where $k \leq n$. Since $\mathrm{Ind}_{\mathrm{P}\mathbb{R}}(\widetilde{R})$ must be equal to n, we see that $k = n$ and that all polynomials q_j $(j = 1, 2, \ldots, n)$ are linear with positive leading coefficients:

$$q_j(z) = \alpha_j z + \beta_j, \qquad \alpha_j > 0, \;\; \beta_j \in \mathbb{R}, \quad j = 1, 2, \ldots, n. \tag{3.24}$$

This follows from the fact that $\sum\limits_{j=1}^{k} \deg q_j = n$ and from (2.14).

Thus, if the function R satisfies (3.8), then we obtain from (3.19) (where α must be nonnegative) and (3.24) that R has a J-fraction expansion (3.9).

4) $\Longrightarrow$ 5) If the function $R(z)$ has a J-fraction expansion (3.9), then

$$R(z) = \frac{q(z)}{p(z)} = -\alpha z + \beta + \frac{f_1(z)}{f_0(z)}, \quad \alpha \geq 0, \quad \beta \in \mathbb{R}, \tag{3.25}$$

where the function $\dfrac{f_1}{f_0}$ has a J-fraction expansion (3.9) and vanishes at ∞. Therefore, starting from the polynomials f_0, f_1, we can construct a sequence $f_0, f_1, \ldots, f_n$ by the Sturm algorithm:

$$f_{j-1}(z) = (\alpha_j z + \beta_j) f_j(z) - f_{j+1}(z), \quad \alpha_j > 0, \quad \beta_j \in \mathbb{R}, \quad j = 0, 1, \ldots, n, \tag{3.26}$$

where $f_{n+1}(z) \equiv 0$, and $\gcd(f_0, f_1) = f_n(z) \equiv 1$, since the polynomials p and q (and, therefore, f_0 and f_1) are coprime by assumption. From (3.26) we have that, for all $\omega \in \mathbb{R}$,

$$\begin{aligned} & f_0(\omega) f_1'(\omega) - f_0'(\omega) f_1(\omega) = -\alpha_1 f_1^2(\omega) + f_1(\omega) f_2'(\omega) - f_1'(\omega) f_2(\omega) \; = \\ & = -\alpha_1 f_1^2(\omega) - \alpha_2 f_2^2(\omega) + f_2(\omega) f_3'(\omega) - f_2'(\omega) f_3(\omega) = \ldots = -\textstyle\sum_{j=1}^{n} \alpha_j f_j^2(\omega) < 0 \end{aligned} \tag{3.27}$$

since all $\alpha_j > 0$ and all $f_j^2(\omega) \geq 0$ and $f_n^2(\omega) > 0$ as well.

Now from (3.25) we obtain that $q(z) = (-\alpha z + \beta) f_0(z) + f_1(z)$ and $p = f_0$. Thus, (3.27) implies

$$p(\omega) q'(\omega) - p'(\omega) q(\omega) = -\alpha f_0^2(\omega) + f_0(\omega) f_1'(\omega) - f_0'(\omega) f_1(\omega) < 0,$$

for any $\omega \in \mathbb{R}$, as required.

Let z be a complex number such that $\operatorname{Im} z \neq 0$. It is easy to see that

$$\operatorname{sign}\left(\operatorname{Im} \frac{-1}{\alpha z + \beta - f(z)}\right) = \alpha \operatorname{sign}(\operatorname{Im} z) - \operatorname{sign}\left(\operatorname{Im} f(z)\right), \quad \alpha \geq 0, \, \beta \in \mathbb{R},$$

where f is any function of a complex variable. Thus, if the function R has a J-fraction expansion (3.9), then we obtain

$$\operatorname{sign}\left(\operatorname{Im} R(z)\right) = \operatorname{sign}\left(\operatorname{Im} \frac{-1}{R(z)}\right) = -\operatorname{sign}\left[\operatorname{Im} z \left(\alpha + \sum_{i=1}^{r} \alpha_i\right)\right] \neq 0,$$

whenever $\operatorname{Im} z \neq 0$. Thus, R necessarily has only *real* zeros and poles[17].

5) $\Longrightarrow$ 6) If the polynomials p and q are real-rooted and satisfy the inequality (3.10), then the condition (3.11) holds for them. From (3.10) it also immediately follows that all zeros of p and q are simple. Otherwise, there must be a *real* number ω such that $p(\omega)q'(\omega) - p'(\omega)q(\omega) = 0$, which contradicts (3.10).

[17] At the same time, this proves the implication 4) $\Longrightarrow$ 1).

Let real numbers ω_1, ω_2 $(\omega_1 < \omega_2)$ be two consecutive (simple) zeros of p. By Rolle's theorem, we have

$$p'(\omega_1)p'(\omega_2) < 0. \tag{3.28}$$

Then from (3.10) it follows that $p'(\omega_1)q(\omega_1) < 0$ and $p'(\omega_2)q(\omega_2) < 0$. Together with (3.28), these inequalities imply $q(\omega_1)q(\omega_2) < 0$. Thus, in the interval (ω_1, ω_2), the polynomial q has an odd number of (simple) zeros. In the same way, one can prove that between any consecutive zeros of q, there is an odd number of zeros of p. Therefore, all zeros of p and q are simple, and between any two consecutive zeros of one of the polynomials there is only one zero of the other polynomial, as required.

6) $\Longrightarrow$ 7) Now let the polynomials p and q have simple real interlacing zeros. If numbers ω_1 and ω_2 $(\omega_1 < \omega_2)$ are some two consecutive zeros of the polynomial p, then by interlacing we have $q(\omega_1)q(\omega_2) < 0$ and the polynomial g defined (3.12) satisfies the inequality $g(\omega_1)g(\omega_2) < 0$. Therefore, all zeros of the polynomial g are real and simple and interlace the zeros of p since $|\deg p - \deg g| \leq 1$.

7) $\Longrightarrow$ 8) Let the polynomial g defined by (3.12) have only real zeros and suppose, without loss of generality, that $\mu \neq 0$. Then the equation

$$p(z)\left(\mu\frac{q(z)}{p(z)} + \lambda\right) = 0 \tag{3.29}$$

has only real solutions for any real λ and $\mu \neq 0$. Therefore, the function $R = \dfrac{q}{p}$ cannot take real values for nonreal z. Otherwise, the equation (3.29) would have nonreal solutions for some real λ and μ. Thus, the function $R = \dfrac{q}{p}$ satisfies (3.13).

The condition (3.11) implies (3.14). Indeed, if a real ω in (3.11) is not a zero of p, then

$$R'(\omega) = \frac{p(\omega)q'(\omega) - p'(\omega)q(\omega)}{p^2(\omega)} < 0.$$

If ω from (3.11) is a zero of the polynomial p, then ω is a pole of the function R'. However, (3.11) shows that, for sufficiently small $\varepsilon > 0$,

$$p(\omega+\varepsilon)q'(\omega+\varepsilon) - p'(\omega+\varepsilon)q(\omega+\varepsilon) < 0,$$

since the function $pg' - p'q$ is continuous and preserves its sign in some vicinity of the point ω. Therefore, $R'(\omega+\varepsilon) < 0$, as required.

8) $\Longrightarrow$ 1) If the function R satisfies (3.13), then it has no complex zeros, since 0 is a real value. Thus, if $\operatorname{Im} z > 0$, then $\operatorname{Im} R(z) \neq 0$, i.e., R is an R-function of positive or negative type. Suppose that $\operatorname{Im} R(z) > 0$ whenever $\operatorname{Im} z > 0$, that is, R is an R-function of positive type. Then the function $F = -R = \dfrac{q}{p}$ is an R-function of negative type, i.e., it satisfies (3.5). But we have already proved that 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4) $\Longrightarrow$ 5), therefore, we have

$$p(\omega)q'(\omega) - p'(\omega)q(\omega) < 0 \quad \text{for all} \quad \omega \in \mathbb{R}.$$

Consequently,

$$F'(\omega) = \frac{p(\omega)q'(\omega) - p'(\omega)q(\omega)}{p^2(\omega)} < 0,$$

for any *real* ω such that $F'(\omega)$ exists. This means that $R'(\omega) = -F'(\omega) > 0$ for any *real* ω such that $R'(\omega)$ exists. This contradicts (3.14). Therefore, R is an R-function of negative type.

4) $\Longleftrightarrow$ 9) Theorem 1.26 guarantees that the function R has a J-fraction expansion (3.9) with all α_j nonzero and *real*, $j = 1, 2, \ldots, n$, if and only if the Hankel minors $D_j(R)$, $j = 1, 2, \ldots, n$, are *nonzero*. But for J-fraction (3.9), the formula (2.22) holds (see also (2.23)). This formula implies that the inequalities (3.16) hold if and only if all $\alpha_j > 0$ for $j = 1, 2, \ldots, n$. $\square$

Remark 3.5. Comparing representations (3.6) and (3.15), one can obtain the following formula:

$$s_i = \sum_{j=1}^{n} \gamma_j \omega_j^i, \qquad i = 0, 1, 2, \dots \tag{3.30}$$

Remark 3.6. From the proof of Theorem 3.4 one can see that a sum of two R-functions of negative type is an R-functions of negative type. Also if $R(z)$ is an R-function of negative type, then the functions $z \mapsto -R(-z)$ and $z \mapsto -\dfrac{1}{R(z)}$ are also R-functions of negative type.

Using the equivalence of conditions 1) and 7) of Theorem 3.4, one can obtain the following simple fact.

Corollary 3.7. *Let p and q be real coprime polynomials satisfying* (3.1)*, $\deg p \geq 2$. If the function $R = q/p$ is an R-function of negative type, then the functions $R_j = q^{(j)}/p^{(j)}$, $j = 1, \dots, \deg p - 1$, are also R-functions of negative type.*

Proof. It suffices to prove that the function R_1 is an R-function. First, consider the case $\deg q = \deg p - 1$:

$$p(z) = \sum_{j=0}^{n} a_j z^{n-j}, \qquad q(z) = \sum_{j=1}^{n} b_j z^{n-j-1}, \quad \text{where} \quad a_0 > 0, \quad b_1 \neq 0, \quad n \geq 2.$$

Then $R(z) \to 0$ as $z \to +\infty$, and, for sufficiently large positive z, $\operatorname{sign} R(z) = \operatorname{sign} q(z)$. Suppose that R is an R-function of negative type. By Theorem 3.4, the polynomial g defined by (3.12) has only real zeros. Therefore, the polynomial g' also has only real zeros, so the function $R_1 = q'/p'$ is an R-function of positive or negative type according to Theorem 3.4 and Remark 3.2

Since R is an R-function of negative type, R is decreasing between its poles (this follows, for example, from (3.6)). Therefore, if a real number ξ is the largest zero of the polynomial p, then R must be positive in the interval $(\xi, +\infty)$, so $R(z) \to +0$ as $z \to +\infty$. Consequently, the polynomial $q(z)$ is positive for sufficiently large positive z, i.e., $b_1 > 0$. Now it is easy to see that

$$p'(z)q''(z) - p''(z)q'(z) \sim -n(n-1)a_0 b_1 z^{2n-4} < 0 \quad \text{as} \quad z \to \pm\infty$$

Hence there exists a real ω such that $p'(\omega)q''(\omega) - p''(\omega)q'(\omega) < 0$, so R_1 is an R-function of negative type by Theorem 3.4.

Let now $\deg q = \deg p$. Then

$$R(z) = \frac{q(z)}{p(z)} = \beta + \frac{h(z)}{p(z)}, \qquad h(z) = q(z) - \beta p(z), \qquad \deg h < \deg p, \tag{3.31}$$

where $\beta \in \mathbb{R}$ and the function h/p is an R-function of negative type by Theorem 3.4 (see (3.6)). Therefore, the function h'/p' is also an R-function of negative type. But from (3.31) it follows that

$$R_1(z) = \frac{q'(z)}{p'(z)} = \beta + \frac{h'(z)}{p'(z)}.$$

Thus, the function R_1 is an R-function of negative type by Theorem 3.4.

Finally, if $\deg q = \deg p + 1$, then we may apply the previous result to the functions $F = -p/q$ and $F_1 = -p'/q'$. ∎

This theorem and Theorem 3.4 immediately imply the following result due to V.A. Markov (see [19, Theorem 9, Chapter 1]).

Theorem 3.8 (V.A. Markov)**.** *If the zeros of two real polynomials p and q are simple, real and interlacing, then the zeros of their derivatives p' and q' also are real, simple and interlacing.*

If a rational function R is an R-function, then, using Theorem 2.14, one can easily find the numbers of negative and positive poles of this function.

Theorem 3.9. *Let a rational function R with exactly r poles be an R-function of negative type and let R have a series expansion* (3.15)*. Then the number r_- of negative poles of R equals*[18]

$$r_- = \mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \ldots, \widehat{D}_k(R)), \tag{3.32}$$

where $k = r - 1$, if $R(0) = \infty$, and $k = r$, if $|R(0)| < \infty$. The determinants $\widehat{D}_j(R)$ are defined in (1.9)*.*

Proof. In fact, Theorem 3.4 states that $\mathrm{V}(1, D_1(R), D_2(R), \ldots, D_r(R)) = 0$ if (and only if) R is an R-function (see (3.16)). Moreover, all poles of R are real and simple, and all residues at those poles are positive (see (3.7)), therefore we get $r_- = \mathrm{Ind}_{-\infty}^{0}(R)$. Thus, the formula (3.32) follows from (2.38)–(2.40), where $\sigma_2 = \sigma_1 = 1$. ∎

It is convenient for us to consider separately the extreme cases of Theorem 3.9.

Corollary 3.10. *Let a rational function R with exactly r poles be an R-function of negative type. All poles of R are negative if and only if*

$$\widehat{D}_{j-1}(R)\widehat{D}_j(R) < 0, \quad j = 1, 2, \ldots, r, \tag{3.33}$$

where $\widehat{D}_0(R) := 1$, and the determinants $\widehat{D}_j(R)$ are defined by (1.9)*.*

Proof. Indeed, Theorem 3.9 implies

$$\mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \ldots, \widehat{D}_r(R)) = r. \tag{3.34}$$

According to the Frobenius rule (2.7), if $\widehat{D}_j(R) = 0$ for some $j \leq r-1$ but $\widehat{D}_{j-1}(R) \neq 0$, then $\operatorname{sign}\widehat{D}_j(R) = \operatorname{sign}\widehat{D}_{j-1}(R)$, that is, $\mathrm{V}(\widehat{D}_{j-1}(R), \widehat{D}_j(R)) = 0$. Therefore, if there are zero determinants $\widehat{D}_j(R)$ in the sequence $(1, \widehat{D}_1(R), \widehat{D}_2(R), \ldots, \widehat{D}_r(R))$, then the equality (3.34) cannot hold. Consequently, all minors $\widehat{D}_j(R)$ for $j = 1, 2, \ldots, r$ are not equal to zero, and from (3.34) we obtain (3.33). ∎

Remark 3.11. The inequalities (3.33) are equivalent to the following inequalities

$$(-1)^j \widehat{D}_j(R) > 0, \quad j = 1, 2, \ldots, r, \tag{3.35}$$

Corollary 3.12. *Let a rational function R with exactly r poles (counting multiplicities) be an R-function of negative type. All poles of R are positive if and only if*

$$\widehat{D}_j(R) > 0, \quad j = 1, 2, \ldots, r, \tag{3.36}$$

where the determinants $\widehat{D}_j(R)$ are defined in (1.9)*.*

Proof. If the function R is an R-function of negative type with positive poles and has a series expansion (3.15), then the function

$$F(z) = -zR(-z) = -\alpha z - \beta + \frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_2}{z^3} - \frac{s_3}{z^4} + \cdots,$$

where $\alpha \geq 0$ and $\beta \in \mathbb{R}$, is an R-function of negative type with negative poles. In fact, since the function R has the form (3.6) with all positive ω_j and γ_j, then F can be represented as follows

$$R(z) = -\alpha z - \beta + \sum_{j=1}^{r} \frac{\gamma_j}{z + \omega_j}, \qquad \alpha \geq 0,\ \beta \in \mathbb{R},\ \omega_j, \gamma_j > 0, \quad j = 1, 2, \ldots, r,$$

[18] Recall that the number $\mathrm{V}(1, \widehat{D}_1(R), \widehat{D}_2(R), \ldots, \widehat{D}_k(R))$ of Frobenius sign changes must be calculated according to Frobenius Rule 2.4.

Therefore, according to Theorem 3.4, the function F is an R-function of negative type with only negative poles. On the other hand, for a fixed integer j $(j \geq 1)$ we have

$$\widehat{D}_j(F) = \begin{vmatrix} -s_1 & s_2 & -s_3 & \dots & (-1)^j s_j \\ s_2 & -s_3 & s_4 & \dots & (-1)^{j+1} s_{j+1} \\ -s_3 & s_4 & -s_5 & \dots & (-1)^{j+2} s_{j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^j s_j & (-1)^{j+1} s_{j+1} & (-1)^{j+2} s_{j+2} & \dots & -s_{2j-1} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^j \end{vmatrix}$$

$$\times \begin{vmatrix} s_1 & s_2 & s_3 & \dots & s_j \\ s_2 & s_3 & s_4 & \dots & s_{j+1} \\ s_3 & s_4 & s_5 & \dots & s_{j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_j & s_{j+1} & s_{j+2} & \dots & s_{2j-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{j-1} \end{vmatrix} = (-1)^{\frac{j(j+1)}{2}} \cdot \widehat{D}_j(R) \cdot (-1)^{\frac{j(j-1)}{2}}$$

$$= (-1)^j \widehat{D}_j(R),$$

Consequently, from these formulæ and the inequalities (3.35) we obtain (3.36). □

3.2 Some classes of infinite Hankel matrices and the finite moment problem on the real axis

Here we connect the characterization of R-functions from Section 3.1 to a particular moment problem, viz., a discrete moment problem on the real line with a measure supported at finitely many points. This problem is quite well known (see, e.g., [3, 91]):

Problem 3.13 (Finite moment problem on $\mathbb{R}$)**.** *Given an infinite sequence of real numbers*

$$(s_0, s_1, s_2, \dots),$$

it is required to determine numbers[19]

$$\gamma_1 > 0, \gamma_2 > 0, \dots, \gamma_n > 0, \qquad \omega_1 < \omega_2 < \dots < \omega_k < 0 \leq \omega_{k+1} < \omega_{k+2} < \dots < \omega_n$$

so that the equations (3.30) *hold:*

$$s_i = \sum_{j=1}^{n} \gamma_j \omega_j^i, \qquad i = 0, 1, 2, \dots \tag{3.30}$$

From Remark 3.5 it follows that the equalities (3.30) are equivalent to the following series representation

$$F(z) := \sum_{j=1}^{n} \frac{\gamma_j}{z - \omega_j} = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \frac{s_3}{z^4} + \cdots \tag{3.37}$$

In this case, the infinite Hankel matrix $S = \|s_{i+j}\|_0^\infty$ is of finite rank n. Thus, our moment problem has a solution if and only if the function F determined by the series (3.37) is an R-function of negative type with r poles and with exactly $k\,(\leq n)$ negative poles. The solution of the moment problem is unique, since the positive numbers γ_j and ω_j $(j = 1, 2, \dots, n)$ are uniquely determined from the expansion (3.37). We will see that Theorems 3.4 and 3.9 provide a solution to this problem in Theorem 3.20 below, with two important special cases provided in Theorem 3.18 and Corollary 3.19. However, before proceeding, we must introduce and discuss some relevant matrix notions of positivity/nonnegativity and sign regularity.

Definition 3.14. An infinite symmetric matrix A of finite rank is called *r-positive definite* if all its principal minors up to order $r(\geq 1)$ (inclusive) are positive. If the rank of the matrix A equals r and A is r-positive definite, then A is called *positive definite*.

[19] For $k = 0$, it is understood that $\omega_1 \geq 0$. Analogously, if $k = n$, we have $\omega_n < 0$.

Definition 3.15. An infinite matrix A of finite rank is called *r-strictly totally positive* if all its minors up to order r (inclusive) are positive. If the rank of the matrix A equals r and A is r-strictly totally positive, then A is called *strictly totally positive*.

Definition 3.16. An infinite matrix is called *totally nonnegative* if all its minors are nonnegative.

If A is a matrix (finite or infinite), then its minor of order $j(\geq 1)$ whose rows are indexed by $i_1, i_2, \dots, i_j$ and whose columns are indexed by $l_1, l_2, \dots, l_j$ is denoted as

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_j \\ l_1 & l_2 & \dots & l_j \end{pmatrix}.$$

Definition 3.17 ([34])**.** An infinite matrix A of finite rank is called *r-sign regular* if all its minors up to order r (inclusive) satisfy the following inequalities:

$$(-1)^{\sum\limits_{k=1}^{j} i_k + \sum\limits_{k=1}^{j} l_k} A\begin{pmatrix} i_1 & i_2 & \dots & i_j \\ l_1 & l_2 & \dots & l_j \end{pmatrix} > 0. \tag{3.38}$$

If, in addition, the rank of the matrix A equals r, then A is called *sign regular*.

With these definitions in place, we are now ready to state and to prove the following theorem.

Theorem 3.18. *A function*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \frac{s_3}{z^4} + \cdots \tag{3.39}$$

is an R-function of negative type and has exactly r poles all of which are positive if and only if the matrix $S = \|s_{i+j}\|_{i,j=0}^{\infty}$ is strictly totally positive of rank r.

Proof. This proof follows the presentation from [36, p. 238].

At first, let us assume that the function R is an R-function of negative type with r poles and all poles are positive. Then by Theorem 3.4 we have

$$R(z) = \sum_{j=1}^{r} \frac{\gamma_j}{z - \omega_j}, \qquad \gamma_j > 0, \quad j = 1, 2, \dots, r, \tag{3.40}$$

where

$$0 < \omega_1 < \omega_2 < \dots < \omega_r. \tag{3.41}$$

According to formulæ (3.30) (with $n = r$), an arbitrary submatrix of the matrix S of order $k\,(\leq r)$ can be represented as follows

$$\begin{aligned} &\begin{pmatrix} s_{i_1+j_1} & \dots & \dots & s_{i_1+j_k} \\ \vdots & \vdots & \vdots & \vdots \\ s_{i_1+j_k} & \dots & \dots & s_{i_k+j_k} \end{pmatrix} = \\ &= \begin{pmatrix} \gamma_1\omega_1^{i_1} & \gamma_2\omega_2^{i_1} & \dots & \gamma_r\omega_r^{i_1} \\ \gamma_1\omega_1^{i_2} & \gamma_2\omega_2^{i_2} & \dots & \gamma_r\omega_r^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1\omega_1^{i_k} & \gamma_2\omega_2^{i_k} & \dots & \gamma_r\omega_r^{i_k} \end{pmatrix} \cdot \begin{pmatrix} \omega_1^{j_1} & \omega_2^{j_2} & \dots & \omega_r^{j_k} \\ \omega_1^{j_1} & \omega_2^{j_2} & \dots & \omega_r^{j_k} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{j_1} & \omega_2^{j_2} & \dots & \omega_r^{j_k} \end{pmatrix}. \end{aligned} \tag{3.42}$$

Therefore,

$$\begin{aligned} &S\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = \begin{vmatrix} s_{i_1+j_1} & \dots & \dots & s_{i_1+j_k} \\ \vdots & \vdots & \vdots & \vdots \\ s_{i_1+j_k} & \dots & \dots & s_{i_k+j_k} \end{vmatrix} = \\ &= \sum_{1\leq\sigma_1<\sigma_2<\ldots<\sigma_k\leq r} \gamma_{\sigma_1}\gamma_{\sigma_2}\cdots\gamma_{\sigma_k} \begin{vmatrix} \omega_{\sigma_1}^{i_1} & \omega_{\sigma_2}^{i_1} & \dots & \omega_{\sigma_k}^{i_1} \\ \omega_{\sigma_1}^{i_2} & \omega_{\sigma_2}^{i_2} & \dots & \omega_{\sigma_k}^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{\sigma_1}^{i_k} & \omega_{\sigma_2}^{i_k} & \dots & \omega_{\sigma_k}^{i_k} \end{vmatrix} \cdot \begin{vmatrix} \omega_{\sigma_1}^{j_1} & \omega_{\sigma_2}^{j_1} & \dots & \omega_{\sigma_k}^{j_1} \\ \omega_{\sigma_1}^{j_2} & \omega_{\sigma_2}^{j_2} & \dots & \omega_{\sigma_k}^{j_2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{\sigma_1}^{j_k} & \omega_{\sigma_2}^{j_k} & \dots & \omega_{\sigma_k}^{j_k} \end{vmatrix} \end{aligned} \tag{3.43}$$

Both determinants here are generalized Vandermonde determinants. Their positivity follows from (3.41), as was proved in [34] (see also [36, p.99 and p.239] or [79, Part V, Chapter 1, Problem 48]). Consequently, any minor of S of order $k(\leq r)$ is positive, since all γ_is are also positive.

Thus, in our case, the matrix S is strictly totally positive and has rank r (the number of poles of the function R), according to Theorem 1.3.

Conversely, if the matrix S is strictly totally positive of rank r, then its leading principal minors $D_j(R)$ are positive up to order r. Since the matrix $S^{(1)} = \|s_{i+j+1}\|_{i,j=0}^{\infty}$ is a submatrix of the matrix S, it is also strictly totally positive and, in particular, its leading principal minors $\widehat{D}_j(R)$ are positive up to order r. From Theorem 3.4 and Corollary 3.12 we obtain that the function R is an R-function of negative type with exactly r poles, which are all positive. $\square$

From this theorem it is easy to obtain the following corollary.

Corollary 3.19. *A function*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \frac{s_3}{z^4} + \cdots$$

is an R-function of negative type and has exactly r poles, all of which are negative, if and only if the matrix $S = \|s_{i+j}\|_{i,j=0}^{\infty}$ is sign regular of rank r.

Proof. The function R is an R-function with only negative poles if and only if the function

$$F(z) = -R(-z) = \frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_2}{z^3} - \frac{s_3}{z^4} + \cdots = \frac{t_0}{z} + \frac{t_1}{z^2} + \frac{t_2}{z^3} + \frac{t_3}{z^4} + \cdots \tag{3.44}$$

is an R-function of negative type with only positive poles (with the same number of poles as the function R).

Let T be the infinite matrix defined by $T := \|t_{i+j}\|_{i,j=0}^{\infty}$. Since $t_j = (-1)^j s_j$ $(j = 1, 2, \ldots)$, we have

$$T = ESE, \tag{3.45}$$

where the infinite matrix E has the form

$$E := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.46}$$

All minors of this matrix except for its principal minors are zero and

$$E\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} = (-1)^{\sum\limits_{l=1}^{k} i_l - k}, \tag{3.47}$$

since $E(j,j) = (-1)^{j-1}$. From the Binet-Cauchy formula (see, e.g., [36, 34]) and from (3.45) we obtain

$$\begin{aligned}
&T\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} \\
&= \sum_{\tau_1<\tau_2<\ldots<\tau_m} \sum_{\varepsilon_1<\varepsilon_2<\ldots<\varepsilon_m} E\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ \tau_1 & \tau_2 & \dots & \tau_k \end{pmatrix} S\begin{pmatrix} \tau_1 & \tau_2 & \dots & \tau_k \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_k \end{pmatrix} E\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} \\
&= E\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} S\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} E\begin{pmatrix} j_1 & j_2 & \dots & j_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} \\
&= (-1)^{\sum\limits_{l=1}^{k} i_l + \sum\limits_{l=1}^{k} j_l - 2k} S\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} = (-1)^{\sum\limits_{l=1}^{k} i_l + \sum\limits_{l=1}^{k} j_l} S\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}.
\end{aligned}$$

Thus,

$$T\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} = (-1)^{\sum\limits_{l=1}^{k} i_l + \sum\limits_{l=1}^{k} j_l} S\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}, \qquad k = 1, 2, \ldots \tag{3.48}$$

This formula and Theorem 3.18 imply the assertion of the corollary. □

Now we are in a position to formulate the solution to Moment Problem 3.13 posed initially. Combined with the results in this section, Theorems 3.4 and 3.9 provide the following solution to this problem:

Theorem 3.20. *The infinite moment problem*

$$s_i = \sum_{j=1}^{n} \gamma_j \omega_j^i, \qquad i = 0, 1, 2, \dots$$

$$\gamma_1 > 0, \gamma_2 > 0, \dots, \gamma_n > 0; \qquad \omega_1 < \omega_2 < \dots < \omega_k < 0 \le \omega_{k+1} < \omega_{k+2} < \dots < \omega_n,$$

where s_i, $i = 0, 1, 2, \dots$, are given real numbers and γ_j and ω_j, $j = 1, 2, \dots, n$, are unknown real numbers, has a solution if and only if the infinite Hankel matrix $S = \|s_{i+j}\|_{i,j=0}^{\infty}$ has rank n, the determinants $D_j(S)$, $j = 1, 2, \dots, n$, defined in (1.6)*, are positive, and*[20]

$$k = \mathrm{V}(1, \widehat{D}_1(S), \widehat{D}_2(S), \dots, \widehat{D}_n(S)),$$

where the determinants $\widehat{D}_j(S)$ $(j = 1, 2, \dots, n)$ are defined by (1.9)*. In that case, the solution is unique.*

3.3 *R*-functions as ratios of polynomials

In this section, we develop determinantal criteria for R-functions involving the Hankel and Hurwitz minors formed from the coefficients of their numerator and denominator.

Suppose that a rational function R is an R-function of negative type, and write

$$R(z) = \frac{q(z)}{p(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots, \tag{3.49}$$

where p and q are real polynomials

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_0, a_1, \dots, a_n \in \mathbb{R},\ a_0 > 0, \tag{3.50}$$

$$q(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n, \quad b_0, b_1, \dots, b_n \in \mathbb{R}. \tag{3.51}$$

Since R is an R-function, we know that $\deg q \ge \deg p - 1$, that is, $b_0^2 + b_1^2 \ne 0$.

Remark 3.21. Generally speaking, the polynomials p and q may have a non-constant greatest common divisor $g = \gcd(p, q)$ of degree $(n - r)$ for some natural $r (< n)$. In this case, one should consider the function R as a ratio of polynomials $\widetilde{p} = p/g$ and $\widetilde{q} = q/g$. Then the number of poles of R equals r.

At first, we describe R-functions in terms of the coefficients of the polynomials p and q. More precisely, we will use the *infinite matrix of Hurwitz type* $H(p, q)$ (see Definition 1.40) and the *finite matrices of Hurwitz type* (Definition 1.42) constructed using the coefficients of polynomials p and q.

Our first two theorems cover the case when $\deg q < \deg p$.

Theorem 3.22. *The function* (3.49)*, where* $\deg q < \deg p$*, is an R-function of negative type with exactly $r\,(\le n)$ poles if and only if*

$$\eta_{2j}(p, q) > 0, \quad j = 1, 2, \dots, r, \tag{3.52}$$

$$\eta_i(p, q) = 0, \quad i > 2r + 1, \tag{3.53}$$

where $\eta_i(p, q)$ are the leading principal minors of the matrix $H(p, q)$, defined by (1.123).

Theorem 3.23. *The function* (3.49)*, where* $\deg q < \deg p$*, is an R-function of negative type with exactly $r\,(\le n)$ poles if and only if*

$$\Delta_{2j-1}(p, q) > 0, \quad j = 1, 2, \dots, r, \tag{3.54}$$

$$\Delta_i(p, q) = 0, \quad i = 2r + 1, 2r + 2, \dots, 2n, \tag{3.55}$$

where $\Delta_i(p, q)$ are the leading principal minors of the Hurwitz matrix $\mathcal{H}_{2n}(p, q)$ defined in (1.125).

[20] Recall that V is the number of Frobenius sign changes. The determinant $D_n(S)$ may be equal to zero, but then $D_{n-1}(S) \ne 0$ in that case by Corollary (1.4).

The next two results cover the case of equal degrees $\deg q = \deg p$:

Theorem 3.24. *The function* (3.49)*, where* $\deg q = \deg p$*, is an R-function of negative type with exactly* $r\,(\leq n)$ *poles if and only if*

$$\eta_{2j+1}(p,q) > 0, \qquad j = 0,1,2,\dots,r, \tag{3.56}$$
$$\eta_i(p,q) = 0, \qquad i > 2r+2, \tag{3.57}$$

where $\eta_i(p,q)$ *are the leading principal minors of the infinite Hurwitz matrix* $H(p,q)$ *defined by* (1.124)*.*

Theorem 3.25. *The function* (3.49)*, where* $\deg q = \deg p$*, is an R-function of negative type with exactly* $r\,(\leq n)$ *poles if and only if*

$$\Delta_{2j}(p,q) \ > \ 0, \qquad j = 1,2,\dots,r, \tag{3.58}$$
$$\Delta_i(p,q) \ = \ 0, \qquad i = 2r+2, 2r+3, \dots, 2n+1, \tag{3.59}$$

where $\Delta_i(p,q)$ *are the leading principal minors of the Hurwitz matrix* $\mathcal{H}_{2n+1}(p,q)$ *defined in* (1.126)*.*

All these results can be easily obtained from Theorems 1.2 and 3.4 and from the formulæ (1.129), (1.131), (1.136), (1.138). Then Theorem 3.9 and the formulæ (1.130), (1.132), (1.137), (1.139) imply

Theorem 3.26. *If the function* (3.49)*, where* $\deg q < \deg p$*, is an R-function of negative type with exactly* $r\,(\leq n)$ *poles, then the number of its positive poles,* $r_+(\leq r)$*, is*

$$r_+ = \mathrm{V}(\eta_1(p,q), \eta_3(p,q), \eta_5(p,q), \dots, \eta_{2k+1}(p,q)) = \mathrm{V}(1, \Delta_2(p,q), \Delta_4(p,q), \dots, \Delta_{2k}(p,q)),$$

where $k = r$*, if* $|R(0)| < \infty$*, and* $k = r-1$*, if* $R(0) = \infty$*.*

Theorem 3.27. *If the function* (3.49)*, where* $\deg q = \deg p$*, is an R-function of negative type with exactly* $r\,(\leq n)$ *poles, then the number of its positive poles,* $r_+(\leq r)$*, is*

$$r_+ = \mathrm{V}(\eta_2(p,q), \eta_4(p,q), \eta_6(p,q), \dots, \eta_{2k+2}(p,q)) = \mathrm{V}(\Delta_1(p,q), \Delta_3(p,q), \Delta_5(p,q), \dots, \Delta_{2k+1}(p,q)),$$

where $k = r$*, if* $|R(0)| < \infty$*, and* $k = r-1$*, if* $R(0) = \infty$*.*

In these theorems, the numbers of sign changes in the sequences of Hurwitz minors must be calculated by the Frobenius Rule (2.7) because of the equalities (1.129)–(1.132) and (1.136)–(1.139) between Hurwitz and Hankel minors.

We next address separately two extreme cases of Theorems 3.26–3.27, when all poles of the function R are either negative or positive:

Corollary 3.28. *The function* (3.49) *is an R-function of negative type with exactly* $r\,(\leq n)$ *poles, all of which are negative, if and only if*

$$\eta_i(p,q) > 0, \qquad j = 1,2,\dots,k,$$
$$\eta_i(p,q) = 0, \qquad i > k,$$

where $k = 2r+1$ *if* $\deg q < \deg p$*, and* $k = 2r+2$ *if* $\deg q = \deg p$*, but* $\eta_i(p,q)$ *are the leading principal minors of the infinite Hurwitz matrix* $H(p,q)$ *defined in* (1.123)*.*

Corollary 3.29. *The function* (3.49) *is an R-function of negative type with exactly* $r\,(\leq n)$ *poles, all of which are negative, if and only if*

$$\Delta_i(p,q) > 0, \qquad i = 1,2,\dots,k,$$
$$\Delta_i(p,q) = 0, \qquad i = k+1, k+2, \dots,$$

where $k = 2r$*, if* $\deg q < \deg p$*, and* $k = 2r+1$*, if* $\deg q = \deg p$*, but* $\Delta_i(p,q)$ *are the leading principal minors of the Hurwitz matrix* $\mathcal{H}_{2n}(p,q)$ *or the matrix* $\mathcal{H}_{2n+1}(p,q)$ *defined in* (1.125)–(1.126)*.*

Corollary 3.30. *The function* (3.49) *is an R-function of negative type with exactly* $r\,(\leq n)$ *poles, all of which are positive, if and only if,*

- *for* $\deg q < \deg p$*, the inequalities* (3.52)–(3.53) *hold and*

$$\eta_{2j-1}(p,q)\eta_{2j+1}(p,q) < 0, \quad j = 1, 2, \dots, r,$$

where $\eta_i(p,q)$ *are the leading principal minors of the matrix* $H(p,q)$ *defined by* (1.123).

- *for* $\deg q = \deg p$*, the inequalities* (3.56)–(3.57) *hold and*

$$\eta_{2j}(p,q)\eta_{2j+2}(p,q) < 0, \quad j = 1, 2, \dots, r,$$

where $\eta_i(p,q)$ *are the leading principal minors of the matrix* $H(p,q)$ *defined by* (1.124).

Corollary 3.31. *The function* (3.49) *is an R-function of negative type with exactly* $r\,(\leq n)$ *poles, all of which are positive, if and only if,*

- *for* $\deg q < \deg p$*, the inequalities* (3.54)–(3.55) *hold and*

$$\Delta_{2j-2}(p,q)\Delta_{2j}(p,q) < 0, \quad j = 1, 2, \dots, r, \quad (\Delta_0(p,q) := 1)$$

where $\Delta_i(p,q)$ *are the leading principal minors of the matrix* $\mathcal{H}_{2n}(p,q)$ *defined in* (1.125).

- *for* $\deg q = \deg p$*, the inequalities* (3.58)–(3.59) *hold and*

$$\Delta_{2j-1}(p,q)\Delta_{2j+1}(p,q) < 0, \quad j = 1, 2, \dots, r,$$

where $\Delta_i(p,q)$ *are the leading principal minors of the matrix* $\mathcal{H}_{2n+1}(p,q)$*, defined in* (1.126).

At last, there is one more way to find the number of negative (or positive) poles of an R-function. This method turns into the famous Lienard-Chipart theorem when applied to the theory of Hurwitz stable polynomials (see [36, Chapter XV, Section 13] Theorem 11). But first, let us introduce (without proof) the remarkable but little-known consequence of the famous Descartes Rule of Signs (see, for example, [79]).

Recall that we denote by $\mathrm{V}^{-}(a_0, a_1, \dots, a_n)$ the number of weak sign changes in a real sequence $(a_0, a_1, \dots, a_n)$, i.e., the number of sign changes with zero elements of the sequence removed.

Theorem 3.32 ([99, 79])**.** *If a real polynomial* $a_0z^n + a_1z^{n-1} + \cdots + a_n$ *has only real roots, then the number of its positive zeros, counting multiplicities, is equal to* $\mathrm{V}^{-}(a_0, a_1, \dots, a_n)$*.*

This fact, together with previous results, implies the following theorem.

Theorem 3.33. *If the real rational function* (3.49) *is an R-function of negative type with exactly* n *poles*[21]*, then the number of its positive poles is equal to* $SC^{-}(a_0, a_1, \dots, a_n)$*. In particular, R has only negative poles if and only if* [22] $a_j > 0$ *for* $j = 1, 2, \dots, n$*, and R has only positive poles if and only if* $a_{j-1}a_j < 0$ *for* $j = 1, 2, \dots, n$*.*

Proof. Indeed, by Theorem 3.4, if the function R defined in (3.49) is an R-function, then the polynomial p has only real roots which are poles of the function R. The number of positive zeros of p (poles of R) equals $\mathrm{V}^{-}(a_0, a_1, \dots, a_n)$, according to Theorem 3.32. ∎

For two extreme classes of R-functions, i.e., those with only positive or only negative poles, we can obtain a few more criteria.

Theorem 3.34. *The function* (3.49)*, where* $\deg q < \deg p = n$*, is an R-function of negative type and has exactly* n *negative poles if and only if one of the following conditions holds*

1) $a_n > 0,\ a_{n-1} > 0,\ \dots,\ a_0 > 0,$ $\qquad \Delta_1(p,q) > 0,\ \Delta_3(p,q) > 0,\ \dots,\ \Delta_{2n-1}(p,q) > 0;$
2) $a_n > 0,\ b_n > 0,\ b_{n-1} > 0,\ \dots,\ b_1 > 0,$ $\qquad \Delta_1(p,q) > 0,\ \Delta_3(p,q) > 0,\ \dots,\ \Delta_{2n-1}(p,q) > 0;$
3) $a_n > 0,\ a_{n-1} > 0,\ \dots,\ a_0 > 0,$ $\qquad \Delta_2(p,q) > 0,\ \Delta_4(p,q) > 0,\ \dots,\ \Delta_{2n}(p,q) > 0;$
4) $a_n > 0,\ b_n > 0,\ b_{n-1} > 0,\ \dots,\ b_1 > 0,$ $\qquad \Delta_2(p,q) > 0,\ \Delta_4(p,q) > 0,\ \dots,\ \Delta_{2n}(p,q) > 0.$

[21] The latter means that $\gcd(p,q) \equiv 1$.

[22] In fact, the coefficients must be simply of the same sign, but we already assumed that $a_0 > 0$ (see (3.50)).

Proof. Condition 1) contains the inequalities (3.54) (with $r = n$). By Theorem 3.23, this is equivalent to the function R being an R-function of negative type with exactly n poles. Now Theorem 3.33 implies that the positivity of the coefficients of p is equivalent to the negativity of poles of R.

Condition 2) also contains the inequalities (3.54) (with $r = n$), which are equivalent to the function R being an R-function of negative type with exactly n poles. By Theorem 3.4, all zeros and poles of R are real and simple.

If the function R has exactly n negative poles, then from Theorem 3.4 it follows that it also has negative zeros (since the zeros and the poles of R are interlacing). Now Theorem 3.32 yields the positivity of all coefficients of the polynomials p and q.

Conversely, if the inequalities $a_n > 0$, $b_n > 0$, $b_{n-1} > 0$, …, $b_1 > 0$ hold, then R has only negative zeros, according to Theorem 3.32. The interlacing of zeros and poles of R implies that R may have at most one nonnegative simple pole, that is, the polynomial p may have at most one nonnegative simple zero. But $a_0 > 0$ by assumption, therefore, $p(z) \to +\infty$ as $z \to +\infty$. Since all zeros of p are simple and real, we have $a_n = p(0) \leq 0$ whenever p has one nonnegative zero. This contradicts the inequality $a_n > 0$. Consequently, p has only negative roots, and R has only negative poles.

If the function R is an R-function of negative type with exactly n negative poles, then Corollary 3.29 and Theorem 3.32 imply condition 3).

Now suppose that condition 3) holds. It contains the inequalities $\Delta_{2j}(p, q) > 0$, $j = 1, 2, \ldots, n$, which are equivalent to the inequalities $(-1)^j \widehat{D}_j(R) > 0$, $j = 1, 2, \ldots, n$, according to (1.137). But we know that $(-1)^j \widehat{D}_j(R) = D_j(F)$, where $F(z) = -zR(z)$, and the positivity of these determinants is equivalent to the function F being an R-function of negative type by Theorem 3.4. From the same theorem it also follows that all zeros of the polynomials $zq(z)$ and $p(z)$ are real simple and interlacing. From Theorem 3.32 and from the inequalities $a_j > 0$, $j = 0, 1, \ldots, n$, we obtain that p has only negative roots. This fact implies that all zeros of q are negative too and that they interlace the zeros of the polynomial p since otherwise the zeros of $zq(z)$ would not interlace the zeros of $p(z)$. Since the zeros of p and q interlace, the function R is an R-function of negative type by Theorem 3.4. It has only negative poles (the zeros of p).

As in the case of condition 3), condition 4) holds if the function R is an R-function of negative type with exactly n negative poles.

If condition 4) holds, then, as before, the function $F(z) = -zR(z)$ is an R-function of negative type and, therefore, the roots of the polynomials $zq(z)$ and $p(z)$ are real and simple and interlace each other. The inequalities $b_j > 0$, $j = 1, 2, \ldots, n$, imply that the polynomial q has only negative zeros. Consequently, $p(z)$ has at most one simple nonnegative zero because of the interlacing with the zeros of $zq(z)$. As above, p cannot have a nonnegative zero since otherwise the inequality $a_n > 0$ cannot hold. Since the polynomial $zq(z)$ has nonpositive zeros but $p(z)$ has only negative zeros and since their zeros interlace each other, the zeros of q and p interlace too. Now Theorem 3.4 implies that R is an R-function of negative type with exactly n negative poles. ∎

In the case $\deg q = \deg p$ we obtain similar criteria. The next theorem addresses the situation when the degrees $\deg p$ and $\deg q$ are equal and all poles of R are negative.

Theorem 3.35. *The function* (3.49), *where* $\deg q = \deg p = n$, *is an* R*-function of negative type with exactly* n *negative poles if and only if one of the following conditions holds*

1) $a_n > 0,\ a_{n-1} > 0,\ \ldots,\ a_0 > 0,\qquad \Delta_2(p, q) > 0,\ \Delta_4(p, q) > 0,\ \ldots,\ \Delta_{2n}(p, q) > 0;$

2) $a_n > 0,\ b_n > 0,\ b_{n-1} > 0,\ \ldots,\ b_0 > 0,\qquad \Delta_2(p, q) > 0,\ \Delta_4(p, q) > 0,\ \ldots,\ \Delta_{2n}(p, q) > 0;$

3) $a_n > 0,\ a_{n-1} > 0,\ \ldots, a_0 > 0,\qquad \Delta_1(p, q) > 0,\ \Delta_3(p, q) > 0,\ \ldots,\ \Delta_{2n+1}(p, q) > 0;$

4) $a_n > 0,\ b_n > 0,\ b_{n-1} > 0,\ \ldots,\ b_0 > 0,\qquad \Delta_1(p, q) > 0,\ \Delta_3(p, q) > 0,\ \ldots,\ \Delta_{2n+1}(p, q) > 0.$

Proof. This theorem can be proved in the same way as Theorem 3.34 using Theorem 3.25 instead of Theorem 3.23 and the equalities (1.139) instead of the equalities (1.137). ∎

To address the situation when all poles of R are positive, it is enough to switch to the function $z \mapsto -R(-z)$ and apply the same methods as in the proof of Theorems 3.34 and 3.35. This yields the following two results, one for the case $\deg q < \deg p$ and the other for the case $\deg q = \deg p$.

Theorem 3.36. *The function* (3.49)*, where* $\deg q < \deg p = n$*, is an* R*-function of negative type and has exactly* n *positive poles if and only if one of the following conditions holds*

1) $$(-1)^n a_n > 0,\ (-1)^{n-1}a_{n-1} > 0,\ \ldots,\ a_0 > 0,$$
$$\Delta_1(p,q) > 0,\ \Delta_3(p,q) > 0,\ \ldots,\ \Delta_{2n-1}(p,q) > 0;$$

2) $$(-1)^n a_n > 0,\ (-1)^{n-1}b_n > 0,\ (-1)^{n-2}b_{n-1} > 0,\ \ldots,\ b_1 > 0,$$
$$\Delta_1(p,q) > 0,\ \Delta_3(p,q) > 0,\ \ldots,\ \Delta_{2n-1}(p,q) > 0;$$

3) $$(-1)^n a_n > 0, (-1)^{n-1}a_{n-1} > 0, \ldots, a_0 > 0,$$
$$-\Delta_2(p,q) > 0,\ \Delta_4(p,q) > 0,\ \ldots,\ (-1)^n\Delta_{2n}(p,q) > 0;$$

4) $$(-1)^n a_n > 0,\ (-1)^{n-1}b_n > 0,\ (-1)^{n-2}b_{n-1} > 0,\ \ldots,\ b_1 > 0,$$
$$-\Delta_2(p,q) > 0,\ \Delta_4(p,q) > 0,\ \ldots,\ (-1)^n\Delta_{2n}(p,q) > 0.$$

Theorem 3.37. *The function* (3.49)*, where* $\deg q = \deg p$*, is an* R*-function of negative type with exactly* n *positive poles if and only if one of the following conditions holds*

1) $$(-1)^n a_n > 0,\ (-1)^{n-1}a_{n-1} > 0,\ \ldots,\ a_0 > 0,$$
$$\Delta_2(p,q) > 0,\ \Delta_4(p,q) > 0,\ \ldots,\ \Delta_{2n}(p,q) > 0;$$

2) $$(-1)^n a_n > 0,\ (-1)^{n-1}b_n > 0,\ (-1)^{n-2}b_{n-1} > 0,\ \ldots,\ b_1 > 0,$$
$$\Delta_2(p,q) > 0,\ \Delta_4(p,q) > 0,\ \ldots,\ \Delta_{2n}(p,q) > 0.$$

3) $$(-1)^n a_n > 0,\ (-1)^{n-1}a_{n-1} > 0,\ \ldots,\ a_0 > 0,$$
$$\Delta_1(p,q) > 0,\ -\Delta_3(p,q) > 0,\ \ldots,\ (-1)^n\Delta_{2n+1}(p,q) > 0;$$

4) $$(-1)^n a_n > 0,\ (-1)^{n-1}b_n > 0,\ (-1)^{n-2}b_{n-1} > 0,\ \ldots,\ b_1 > 0,$$
$$\Delta_1(p,q) > 0,\ -\Delta_3(p,q) > 0,\ \ldots,\ (-1)^n\Delta_{2n+1}(p,q) > 0;$$

3.4 R-functions with a Stieltjes continued fraction expansion

We now provide criteria for R functions based on the coefficients of their Stieltjes continued fractions.

Assume that our function (3.49) with exactly $r\,(\leq n)$ poles has a Stieltjes continued fraction expansion

$$R(z) = c_0 + \cfrac{1}{c_1 z + \cfrac{1}{c_2 + \cfrac{1}{c_3 z + \cfrac{1}{\ddots + \cfrac{1}{T}}}}}, \quad c_j \in \mathbb{R}, \quad c_j \neq 0, \tag{3.60}$$

where

$$T = \begin{cases} c_{2r} & \text{if } \ |R(0)| < \infty, \\ c_{2r-1}z & \text{if } \ R(0) = \infty, \end{cases} \tag{3.61}$$

and $c_0 = s_{-1}$, $c_0 \neq 0$ if and only if $\deg p = \deg q (= n)$. From Theorem 1.46 it follows that the inequalities (1.106)–(1.107) hold for the function R. At the same time the coefficients c_i in (3.60) can be found by the formulæ (1.114)–(1.115) (see (1.116)).

Theorems 3.4 and 2.15 yield

Theorem 3.38. *Let the function* (3.49) *with exactly* $r\,(\leq n = \deg p)$ *poles have a Stieltjes continued fraction expansion* (3.60)–(3.61)*. The function* R *is an* R*-function of negative type if and only if*

$$c_{2j-1} > 0, \qquad j = 1, 2, \ldots, r. \tag{3.62}$$

Then the number of negative poles is equal to the number of positive coefficients c_{2j}, $j = 1, 2, \ldots, k$, *where* $k = r$, *if* $|R(0)| < \infty$, *and* $k = r - 1$, *if* $R(0) = \infty$.

Note that R-functions with only positive or only negative poles have Stieltjes continued fraction expansions, according to Corollaries 3.10 and 3.12 and Theorem 1.46. Theorem 3.38 has the following corollaries.

Corollary 3.39. *A rational function* R *with exactly* r *poles is an* R*-function of negative type with all positive poles if and only if* R *has a Stieltjes continued fraction expansion* (3.60)–(3.61)*, where inequalities* (3.62) *hold and where*

$$c_{2j} < 0, \qquad j = 1, 2, \ldots, r.$$

Corollary 3.40. *A rational function* R *with exactly* r *poles is an* R*-function of negative type with all positive poles, except for one at* 0*, if and only if* R *has a Stieltjes continued fraction expansion* (3.60)–(3.61)*, where inequalities* (3.62) *hold and where*

$$c_{2j} < 0, \qquad j = 1, 2, \ldots, r - 1.$$

Corollary 3.41 (Markov, Stieltjes [67, 93, 94, 95, 96])**.** *A rational function* R *with exactly* r *poles is an* R*-function of negative type with all negative poles if and only if* R *has a Stieltjes continued fraction expansion* (3.60)–(3.61)*, where*

$$c_i > 0, \qquad i = 1, 2, \ldots, 2r. \tag{3.63}$$

Corollary 3.42. *A rational function* R *with exactly* r *poles is an* R*-function of negative type with all negative poles, except for one at* 0*, if and only if* R *has a Stieltjes continued fraction expansion* (3.60)–(3.61)*, where*

$$c_i > 0, \qquad i = 1, 2, \ldots, 2r - 1. \tag{3.64}$$

In the sequel we use the following well-known result.

Theorem 3.43 (Aisen, Edrei, Schoenberg, Whitney, [1, 2, 25, 54])**.** *The polynomial*

$$g(z) = g_0 z^l + g_1 z^{l-1} + \cdots + g_l$$

has only nonpositive zeros if and only if its Toeplitz matrix $\mathcal{T}(g)$ *defined by* (1.128) *is totally nonnegative.*

The functions whose Taylor coefficients generate totally nonnegative Toeplitz matrices of the form $\mathcal{T}(g)$ was introduced by Schoenberg [85, 86] and also studied by Edrei [27, 26, 5].

We next prove a criterion of total nonnegativity of infinite Hurwitz matrices. Previously, only one direction, i.e., the fact that the infinite Hurwitz matrix of a quasi-stable polynomial[23] is totally nonnegative (see [6, 57, 46, 55]), was known. The necessary and sufficient condition was known only for finite Hurwitz matrix of Hurwitz stable polynomials.

Theorem 3.44 (**Total Nonnegativity of the Hurwitz Matrix**)**.** *The following are equivalent:*

1) *The polynomials* p *and* q *defined by* (3.50)–(3.51) *have only nonpositive zeros*[24]*, and the function* $R = q/p$ *is either an* R*-function of negative type or identically zero.*

2) *The infinite matrix of Hurwitz type* $H(p, q)$ *defined by* (1.123)–(1.124) *is totally nonnegative.*

[23] The quasi-stable polynomials are polynomials with zeros in the closed left half-plane of the complex plane.
[24] Here we include the case when $q(z) \equiv 0$.

Proof. If condition 1) holds and $R(z) \not\equiv 0$, then, according to Corollaries 3.41–3.42, the function R has a Stieltjes fraction expansion (3.60)–(3.61) and satisfies the inequalities (3.63) or (3.64), where $r\,(\leq n)$ is the number of poles of the function $R = q/p$. According to Theorem 1.46, the matrix $H(p, q)$ has a factorization (1.140) or (1.141), where all matrices $J(c_j)$ are totally nonnegative by inspection since all c_j are positive. However, by assumption, all zeros of p and q are nonpositive, therefore all zeros of $g = \gcd(p, q)$ are also nonpositive. Now from Theorem 3.43 we obtain that the matrix $\mathcal{T}(g)$ is totally nonnegative. Since the matrix $H(0, 1)$ is trivially totally nonnegative, the matrix $H(p, q)$ is totally nonnegative as a product of totally nonnegative matrices (see [34]).

Let condition 1) hold and $R(z) \equiv 0$. Then instead of the factorizations (1.140)–(1.141), we obtain $H(p, q) = H(0, 1)\mathcal{T}(p)$. So in this case $H(p, q)$ is also totally nonnegative.

Conversely, let the matrix $H(p, q)$ be totally nonnegative and $q(z) \equiv 0$. In this case, $\mathcal{T}(p)$ is also totally nonnegative as a submatrix of $H(p, q)$. By Theorem 3.43, p has only nonpositive zeros, as required.

Let now the matrix $H(p, q)$ be totally nonnegative and let $q(z) \not\equiv 0$. All submatrices of $H(p, q)$ are also totally nonnegative. Since matrices $\mathcal{T}(p)$ and $\mathcal{T}(q)$ are submatrices of $H(p, q)$ constructed using all its columns and all even or all odd rows, they are totally nonnegative. Thus, according to Theorem 3.43, the polynomials p and q have only nonpositive zeros. Let g be the greatest common divisor of p and q, so that $p = \widetilde{p}g$ and $q = \widetilde{q}g$. Then Theorems 1.43 and 3.43 imply that $H(p, q) = H(\widetilde{p}, \widetilde{q})\mathcal{T}(g)$, where the matrix $\mathcal{T}(g)$ is totally nonnegative since g has only nonpositive zeros.

First, assume that $\deg p < \deg q$, that is, $R(\infty) = 0$, and use notation

$$\begin{aligned} p(z) \ &=: \ f_0(z) \ =: \ a_0^{(0)} z^n + a_1^{(0)} z^{n-1} + \cdots + a_n^{(0)}, \\ q(z) \ &=: \ f_1(z) \ =: \ a_0^{(1)} z^{n-1} + a_1^{(1)} z^{n-2} + \cdots + a_{n-1}^{(1)}, \end{aligned}$$

where $a_0^{(0)} = a_0 > 0$ by assumption (see (3.50)). From the total nonnegativity of the matrix $H(p, q)$, which is the same as $H(f_0, f_1)$, we have $a_i^{(0)} \geq 0$, $a_{i-1}^{(1)} \geq 0$, $i = 1, 2, \ldots, n$.

Show that $\deg f_1 = n - 1$. To this end, let us introduce notation $H_0 := H(f_0, f_1)$ and suppose that $a_0^{(1)} = a_1^{(1)} = \cdots = a_{j-1}^{(1)} = 0$ and[25] $a_j^{(1)} > 0$ for some integer $1 < j \leq n - 1$. In this case, we have

$$H_0 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & j+2 \end{pmatrix} = \begin{vmatrix} a_0^{(0)} & a_1^{(0)} & a_{j+1}^{(0)} \\ 0 & 0 & a_j^{(1)} \\ 0 & a_0^{(0)} & a_j^{(0)} \end{vmatrix} = -\left(a_0^{(0)}\right)^2 a_j^{(1)} < 0.$$

This inequality contradicts the total nonnegativity of the matrix H_0. Thus, we obtain that $a_0^{(1)} > 0$ and, therefore, the polynomial f_1 is of exact degree $n - 1$.

Now we can perform the first step of the algorithm (1.119)–(1.120) to get the next polynomial f_2:

$$f_2(z) = f_0(z) - c_1 z f_1(z) = a_0^{(2)} z^{n-1} + a_1^{(2)} z^{n-2} + \cdots + a_{n-1}^{(2)}, \tag{3.65}$$

where $c_1 = \dfrac{a_0^{(0)}}{a_0^{(1)}} > 0$. As in the proof of Theorem 1.46, we obtain the factorization

$$H_0 = J(c_1) H_1, \tag{3.66}$$

where $H_1 := H(f_2, f_1)$ and the matrix $J(c_1)$ is defined as in (1.142). If $f_2(z) \equiv 0$, then $f_1 = \gcd(f_0, f_1)$ and the function $R(z) = \dfrac{1}{c_1 z}$ is an R-function of negative type. So, in this case, the theorem is proved. If $f_2(z) \not\equiv 0$, then from (3.65) and from the total nonnegativity of the matrix $H(f_0, f_1)$ it follows that

$$a_i^{(2)} = \frac{1}{a_0^{(1)}} \cdot \begin{vmatrix} a_0^{(1)} & a_{i+1}^{(1)} \\ a_0^{(0)} & a_{i+1}^{(0)} \end{vmatrix} = \frac{1}{a_0^{(0)} a_0^{(1)}} \cdot H_0 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & i+3 \end{pmatrix} \geq 0, \quad i = 0, 1, \ldots, n-1.$$

Suppose that $\deg f_2 < n - 1$. Then there exists an integer j, $1 \leq j \leq n - 1$, such that the coefficients $a_0^{(2)} = a_1^{(2)} = \ldots = a_{j-1}^{(2)} = 0$ and $a_j^{(2)} > 0$. Then from (3.66) we obtain

$$H_0 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & j+3 \end{pmatrix} = c_1 \cdot H_1 \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & j+3 \end{pmatrix} = c_1 \left(a_0^{(1)}\right)^2 \cdot \begin{vmatrix} 0 & a_j^{(2)} \\ a_0^{(1)} & a_j^{(1)} \end{vmatrix} < 0,$$

[25] This is possible since $q(z) \not\equiv 0$ by assumption.

which contradicts the total nonnegativity of the matrix H_0. So, the coefficient $a_0^{(2)}$ must be positive and, therefore, $\deg f_2 = n-1$. Thus, we can perform the next step of the algorithm (1.119)–(1.120) to obtain the next polynomial f_3:

$$f_3(z) = f_1(z) - c_2 f_2(z) = a_0^{(3)} z^{n-2} + a_1^{(3)} z^{n-3} + \cdots + a_{n-2}^{(3)}, \tag{3.67}$$

where $c_2 = \dfrac{a_0^{(1)}}{a_0^{(2)}} > 0$. The formulæ (3.66)–(3.67) imply

$$a_i^{(3)} = \frac{1}{a_0^{(2)}} \cdot \begin{vmatrix} a_0^{(2)} & a_{i+1}^{(2)} \\ a_0^{(1)} & a_{i+1}^{(1)} \end{vmatrix} = \frac{1}{a_0^{(2)} \left(a_0^{(1)}\right)^2} \cdot H_1 \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & i+4 \end{pmatrix} = \frac{1}{a_0^{(2)} \left(a_0^{(1)}\right)^2} \cdot \frac{1}{c_1} \cdot H_0 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & i+4 \end{pmatrix}$$

$$= \frac{1}{a_0^{(0)} a_0^{(1)} a_0^{(2)}} \cdot H_0 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & i+4 \end{pmatrix} \geq 0, \qquad i = 0, 1, \ldots, n-2.$$

For the matrix $H(f_0, f_1)$, we have the factorization

$$H_0 = J(c_1) J(c_2) H_2,$$

where $H_2 = H(f_2, f_1)$.

Now let us assume that we have a sequence of polynomials $f_0, f_1, \ldots, f_m$ $(1 \leq m < 2r)$ constructed from the polynomials f_0 and f_1 by the algorithm (1.119)–(1.120), that is,

$$\begin{aligned} f_{2i}(z) &= f_{2i-2}(z) - c_{2i-1} z f_{2i-1}(z), \qquad i = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \\ f_{2i+1}(z) &= f_{2i-1}(z) - c_{2i} f_{2i}(z), \qquad i = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil - 1, \end{aligned} \tag{3.68}$$

where $\deg f_{2i-1} = \deg f_{2i} = n - i$. Let us also assume that all the coefficients $a_i^{(j)}$ of each polynomial f_j are nonnegative. Then the coefficients c_j in (3.68) are positive and determined by the formula

$$c_j = \frac{a_0^{(j-1)}}{a_0^{(j)}} > 0, \qquad j = 1, 2, \ldots, m-1. \tag{3.69}$$

Moreover, if we set

$$H_{m-1} := \begin{cases} H(f_{m-1}, f_m) & \text{if } m \text{ is odd}, \\ H(f_m, f_{m-1}) & \text{if } m \text{ is even}, \end{cases}$$

then the matrix H_0 has the following factorization

$$H_0 = J(c_1) J(c_2) \cdots J(c_{m-1}) H_{m-1}, \tag{3.70}$$

as it follows from the proof of Theorem 1.46.

Let us perform the next step of the algorithm (1.119)–(1.120) and obtain the next polynomial f_{m+1}:

$$f_{m+1}(z) = \begin{cases} f_{m-1}(z) - c_m z f_m(z) & \text{if } m \text{ is odd}, \\ f_{m-1}(z) - c_m f_m(z) & \text{if } m \text{ is even}, \end{cases} \tag{3.71}$$

and denote the coefficients of f_{m+1} by $a_i^{(m+1)}$. If $f_{m+1}(z) \equiv 0$, then $m = 2r-1$ or $m = 2r-2$ (r is the number of poles of the function $R = \dfrac{f_1}{f_0}$) and $f_m = \gcd(f_0, f_1)$. Now the formula (3.69) and Corollaries 3.41 and 3.42 show that R is an R-function of negative type with nonnegative poles, which completes the proof in this case. If $f_{m+1}(z) \not\equiv 0$, then it follows from (3.70)–(3.71) and from the total nonnegativity of the matrix H_0 that

$$a_i^{(m+1)} = \frac{1}{a_0^{(0)} a_0^{(1)} \cdots a_0^{(m)}} \cdot H_0 \begin{pmatrix} 1 & 2 & \ldots & m+1 & m+2 \\ 1 & 2 & \ldots & m+1 & i+m+2 \end{pmatrix} \geq 0, \quad i = 0, 1, 2, \ldots, n - \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

We will show that $a_0^{(m+1)} > 0$. In fact, if we suppose that it is not true, then there exists a number j $(1 \leq j \leq n - \left\lfloor \dfrac{m}{2} \right\rfloor - 1)$ such that $a_0^{(m+1)} = a_1^{(m+1)} = \cdots = a_{j-1}^{(m+1)} = 0$ and $a_j^{(m+1)} > 0$. Then (3.70) yields

$$H_0 \begin{pmatrix} 1 & 2 & \dots & m+2 & m+3 \\ 1 & 2 & \dots & m+2 & j+m+2 \end{pmatrix} = c_1 c_2^2 \cdots c_m^m \left(a_0^{(m)}\right)^{m+1} \cdot \begin{vmatrix} 0 & a_j^{(m+1)} \\ a_0^{(m)} & a_j^{(m)} \end{vmatrix} < 0.$$

This inequality contradicts the total nonnegativity of the matrix H_0. Consequently, $a_0^{(m+1)} > 0$ and we can run the next step of the algorithm (1.119)–(1.120) to obtain the next polynomial f_{m+2}.

Thus, step by step we construct a sequence of positive numbers $c_1, c_2, \dots, c_k$, where $k = 2r$ if $|R(0)| < \infty$, and $k = 2r - 1$ otherwise. These numbers are exactly the coefficients of the Stieltjes continued fraction expansion of the function R. In this case, R is an R-function of negative type with nonpositive poles according to Corollaries 3.41 and 3.42, as required.

If $\deg p = \deg q$, then $R(\infty) = c_0 = \dfrac{b_0}{a_0} > 0$, since $H(p,q)$ is totally nonnegative. If we denote $f_{-1} = p$ and $f_0 = q$ and run the algorithm (1.119)–(1.120) as before, then we obtain that the function $R = p/q$ is an R-function of negative type with nonpositive poles. ∎

Remark 3.45. Note that the easier direction 1) $\Longrightarrow$ 2) of Theorem 3.44 was proved in [28, Proposition 3.22] as a generalization of results due to Asner, Kemperman and Holtz [6, 57, 46], but the more complicated implication 2) $\Longrightarrow$ 1) appears to be new.

For finite matrices of Hurwitz type, there is no criterion analogous to Theorem 3.44. However, the following two theorems can be derived as straightforward consequences of Theorem 3.44.

Theorem 3.46. *If the polynomials p and q defined in* (3.50)–(3.51) *have only nonpositive zeros, and the function $R = q/p$ is either an R-function of negative type or identically zero, then the finite matrix of Hurwitz type $\mathcal{H}_k(p,q)$ is totally nonnegative. Here $k = 2n$ if $\deg q < \deg p$, and $k = 2n+1$ if $\deg q = \deg p$.*

Theorem 3.47 (**Total Nonnegativity of the Finite Hurwitz Matrix**)**.** *Given the polynomials p and q defined in* (3.50)–(3.51)*, the function $R = q/p$ is an R-function with exactly n negative poles if and only if the finite matrix of Hurwitz type $\mathcal{H}_k(p,q)$ is nonsingular and totally nonnegative. Here $k = 2n$ if $\deg q < \deg p$, and $k = 2n+1$ if $\deg q = \deg p$.*

Proof. Indeed, if the function R is an R-function with exactly n *negative poles* poles, then the function $zR(z)$ also has exactly n poles, so the corresponding Hankel minor $\widehat{D}_n(R)$ is nonzero according to Theorem 1.3. By Theorem 1.45, this means that $\det \mathcal{H}_k(p,q) \neq 0$, so the matrix $\mathcal{H}_k(p,q)$ is nonsingular. But $\mathcal{H}_k(p,q)$ is totally nonnegative as a submatrix of the totally nonnegative matrix $H(p,q)$ (see Theorem 3.44).

Conversely, let the matrix $\mathcal{H}_k(p,q)$ be nonsingular and totally nonnegative. The nonsingularity of the matrix $\mathcal{H}_k(p,q)$ implies that the function $R = q/p$ has exactly $n = \deg p$ poles, that is, the polynomials p and q are coprime. Now by the same methods as those used in the proof of Theorem 3.44, we can show that the total nonnegativity of the matrix $\mathcal{H}_k(p,q)$ implies that the function R has a Stieltjes continued fraction expansion (3.60)–(3.61) with positive coefficients, which is equivalent to the function $R = q/p$ being an R-function with negative poles, according to Corollary 3.41. ∎

In the particular case when p and q are the even and odd parts of some polynomial, Theorem 3.47 was first established by Asner [6].

Let the polynomials p and q be defined in (3.50)–(3.51) and let $k = 2n$ if $\deg q < \deg p$, and $k = 2n+1$ if $\deg q = \deg p$. In the same way as in Theorem 1.43, one can show the following: if the polynomials have a common divisor g of degree l such that $p = \widetilde{p}g$ and $q = \widetilde{q}g$, then

$$\mathcal{H}_k(p,q) = \mathcal{H}_k(\widetilde{p},\widetilde{q})\mathcal{T}_k(g), \tag{3.72}$$

where the matrix $\mathcal{H}_k(\widetilde{p},\widetilde{q})$ is the $k \times k$ principal submatrix of the infinite matrix $H(\widetilde{p},\widetilde{q})$ indexed by rows (and columns) 2 through $k+1$, and the matrix $\mathcal{T}_k(g)$ is the $k \times k$ leading principal submatrix of the matrix $\mathcal{T}(g)$ defined in (1.128).

If the matrix $\mathcal{H}_k(p,q)$ is singular and totally nonnegative, then $\mathcal{H}_k(p,q)$ can be represented as in (3.72), where the polynomials $\widetilde{p}$ and $\widetilde{q}$ have only nonpositive zeros and $\widetilde{R} = \widetilde{q}/\widetilde{p}$ is either an R-function or

$\widetilde{R}(z) \equiv 0$, but the polynomial $g = \gcd(p, q)$ has no nonpositive zeros or $g(z) \equiv \text{const} \neq 0$. This factorization of the totally nonnegative matrix $\mathcal{H}_k(p, q)$ is possible, for example, if all minors of order $\leq k$ of the infinite matrix $\mathcal{T}(g)$ are nonnegative.

Remark 3.48. If all minors of order $\leq k$ of the infinite matrix $\mathcal{T}(g)$ are nonnegative, then the sequence of the coefficients of the polynomial g is called *k-times positive* or *k-positive.* If $\mathcal{T}(g)$ is totally nonnegative, then the sequence of the coefficients of the polynomial g is called *totally positive* [87, 88]. The functions generating k-positive (totally positive) sequences are usually denoted by PF_k (PF_∞).

Based on the results above, we make the following conjecture.

Conjecture 3.49. *Given two polynomials p and q defined in (3.50)–(3.51), the finite matrix*[26] *$\mathcal{H}_k(p, q)$ is totally nonnegative if and only if $\widetilde{p}$ and $\widetilde{q}$ have only nonpositive zeros and $\widetilde{R} = \widetilde{q}/\widetilde{p}$ is either an R-function or $\widetilde{R}(z) \equiv 0$, and the polynomial $g = \gcd(p, q)$ has no real zeros and belongs to the class $PF_{k-\deg g}$.*

Theorems 3.44 and 3.46 imply the following corollaries.

Corollary 3.50. *The following conditions are equivalent:*

1) *The polynomials p and q defined by (3.50)–(3.51) are coprime and have only negative zeros, and the function $R = q/p$ is an R-function of negative type.*

2) *The infinite matrix of Hurwitz type $H(p, q)$ defined by (1.123)–(1.124) is totally nonnegative and $\eta_k(p, q) > 0$, where $k = 2n$ if $\deg q < \deg p$, and $k = 2n + 1$ if $\deg q = \deg p$.*

Corollary 3.51. *The following conditions are equivalent:*

1) *The polynomials p and q defined by (3.50)–(3.51) are coprime and have only nonpositive roots, and the function $R = q/p$ is an R-function of negative type.*

2) *The finite matrix of Hurwitz type $H_k(p, q)$ is totally nonnegative of rank $k - 1$, where $k = 2n$ if $\deg q < \deg p$, and $k = 2n + 1$ if $\deg q = \deg p$.*

Sometimes it is convenient to use the inverse indexing of polynomial coefficients. We now state and prove a result analogous to Theorem 3.44, using this alternative ordering of coefficients.

Corollary 3.52. *For the polynomials*

$$
\begin{aligned}
g(z) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n, \quad a_n > 0, \quad a_0 \neq 0, && (3.73)\\
h(z) &= b_1 + b_2 z + \cdots + b_{n-1} z^{n-2} + b_n z^{n-1}, \qquad b_n > 0, && (3.74)
\end{aligned}
$$

the following conditions are equivalent:

1) *The polynomials g and h have only negative zeros, and the function $R = h/g$ is an R-function of negative type.*

2) *The infinite matrix*

$$
H_\infty(g, h) := \begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\
0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\
0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\
0 & 0 & b_1 & b_2 & b_3 & b_4 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

   is totally nonnegative.

Proof. Let the polynomials g and h have only negative zeros such that $R = h/g$ is an R-function of negative type with exactly $m\,(\leq n)$ poles. Therefore, by Theorem 3.4, R can be represented as follows

$$
R(z) = \frac{b_1 + b_2 z + \cdots + b_{n-1} z^{n-2} + b_n z^{n-1}}{a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n} = \sum_{j=1}^{m} \frac{\alpha_j}{z + \lambda_j}, \quad \alpha_j, \lambda_j > 0.
$$

[26] Here $k = 2n$ if $\deg q < \deg p$, and $k = 2n + 1$ if $\deg q = \deg p$.

All zeros of this function are also negative. Therefore, $b_1 \neq 0$. Consider the function

$$\widetilde{R}(z) = \frac{1}{z} R\left(\frac{1}{z}\right) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_{n-1} z + b_n}{a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n} = \sum_{j=1}^{m} \frac{\alpha_j/\lambda_j}{z + 1/\lambda_j}, \quad \alpha_j, \lambda_j > 0. \tag{3.75}$$

Thus, if R is an R-function of negative type with negative poles, then $\widetilde{R}$ is also an R-function of negative type with negative poles. It is easy to show that the converse statement is also valid. Now by Theorem 3.44 and by (3.75), we obtain the equivalence of the conditions 1) and 2) of the theorem. ∎

In the same way, one can prove the following corollary.

Corollary 3.53. *For the polynomials*

$$\begin{aligned} g(z) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n, \quad a_n > 0, & (3.76)\\ h(z) &= b_0 + b_1 z + b_2 z^2 + \cdots + b_{n-1} z^{n-1} + b_n z^n, \quad b_n > 0, \quad b_0 \neq 0, & (3.77) \end{aligned}$$

the following conditions are equivalent:

1) *The polynomials g and h have only negative zeros, and the function $R = g/h$ is an R-function of negative type.*

2) *The infinite matrix*

$$H_\infty(g,h) = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative.

Corollaries 3.52–3.53 imply the following two theorems, which, in fact, define a class of interlacing preservers (see [28] and references there) and imply one theorem originally proved by Pólya.

Theorem 3.54. *Let the polynomials g and h defined in* (3.73)–(3.74) *have only negative zeros, and let the function $R = h/g$ be an R-function of negative type. Given any two positive integers r and l such that $rl \leq n < (l+1)r$, the polynomials*

$$\begin{aligned} g_{r,l}(z) &:= a_0 + a_r z + a_{2r} z^2 + a_{3r} z^3 + \cdots + a_{rl} z^l, & (3.78)\\ h_{r,l}(z) &:= b_r + b_{2r} z + b_{3r} z^2 + \cdots + b_{rl} z^{l-1} & (3.79) \end{aligned}$$

have only negative zeros, and the function $R_{r,l} = h_{r,l}/g_{r,l}$ is an R-function of negative type.

Proof. Indeed, if the polynomials g and h have only negative zeros, and if the function $R = h/g$ is an R-function of negative type, then by Corollary 3.53, the matrix $H_\infty(g,h)$ is totally nonnegative. Then all its submatrices are totally nonnegative. In particular, the following infinite submatrix whose columns are indexed by $1, r+1, 2r+1, 3r+1, \dots$ and rows are indexed by $1, 2r+2, 4r+3, 6r+4, \dots$

$$H_\infty(g_{r,l}, h_{r,l}) = \begin{pmatrix} a_0 & a_r & a_{2r} & a_{3r} & a_{4r} & a_{5r} & \dots \\ 0 & b_r & b_{2r} & b_{3r} & b_{4r} & b_{5r} & \dots \\ 0 & a_0 & a_r & a_{2r} & a_{3r} & a_{4r} & \dots \\ 0 & 0 & b_r & b_{2r} & b_{3r} & b_{4r} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative. Now by Corollary 3.53, the polynomials (3.78)–(3.79) have only negative zeros, and $R_{r,l} = h_{r,l}/g_{r,l}$ is an R-function of negative type, as required. ∎

The following theorem can be proved in the same fashion.

Theorem 3.55. *Let the polynomials g and h defined in* (3.76)–(3.77) *have only negative zeros, and let the function $R = g/h$ be an R-function of negative type. Given any two positive integers r and l such that $rl \leq n < (l+1)r$, the polynomials*

$$\begin{aligned} g_{r,l}(z) &:= a_0 + a_r z + a_{2r} z^2 + a_{3r} z^3 + \cdots + a_{rl} z^l, \\ h_{r,l}(z) &:= b_0 + b_r z + b_{2r} z^2 + b_{3r} z^3 + \cdots + b_{rl} z^l \end{aligned}$$

have only negative zeros, and the function $R_{r,l} := g_{r,l}/h_{r,l}$ is an R-function of negative type.

Theorems 3.54–3.55 imply the following result of Pólya [78, p. 319] (also implied by Theorem 3.43).

Corollary 3.56. *If the polynomial*

$$g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n, \quad a_n > 0, \tag{3.80}$$

has only negative zeros, then for any positive integers r and l satisfying $rl \leq n < (l+1)r$, the polynomial

$$g_{r,l}(z) = a_0 + a_r z + a_{2r} z^2 + \cdots + a_{rl} z^l$$

also has only negative zeros.

An analogous result can be obtained for polynomials with only positive zeros.

Corollary 3.57. *Let the polynomial* (3.80) *have only positive zeros. Then for any positive integers r and l satisfying $rl \leq n < (l+1)r$, the polynomial* (3.56) *also has only positive (negative) zeros whenever r is odd (even).*

4 The number of distinct real zeros of polynomials. Polynomials with all real zeros

In this section we present a sample application of the theory developed in the previous sections. Using those methods, we analyze the distribution of zeros of real polynomials with respect to the real and the imaginary axes.

We should note that polynomials with all real roots have been studied in control theory, where this property is referred to as *aperiodicity*. Among the many relevant papers we note the work of Jury, Meerov, Fuller and Datta [33, 31, 32, 30, 68, 21, 13] containing special cases of our results, albeit derived using mostly different methods.

4.1 The number of distinct positive, negative and non-real zeros of polynomials. Stieltjes continued fractions of the logarithmic derivative

Consider a real polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad a_1, \ldots, a_n \in \mathbb{R},\ a_0 > 0,\ n \geq 1. \tag{4.1}$$

We denote its logarithmic derivative by $L(z)$:

$$L(z) := \frac{d \log(p(z))}{dz} = \frac{p'(z)}{p(z)} = \frac{n a_0 z^{n-1} + (n-1) a_1 z^{n-2} + \cdots + a_{n-1}}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}.$$

If

$$p(z) = a_0 (z - \lambda_1)^{n_1} (z - \lambda_2)^{n_2} \ldots (z - \lambda_m)^{n_m}, \qquad n_1 + n_2 + \cdots + n_m = n,$$

where n_j is the multiplicity of the zero λ_j $(j = 1, 2, \ldots, m)$, then the logarithmic derivative of the polynomial p has the following form

$$L(z) = \sum_{j=1}^{m} \frac{n_j}{z - \lambda_j}. \tag{4.2}$$

Moreover, if we expand the function L into its Laurent series at ∞

$$L(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \frac{s_3}{z^4} + \cdots, \tag{4.3}$$

then the coefficients s_j are the Newton sums of the polynomial p (see, for instance, [36]):

$$s_k = \sum_{j=1}^{m} n_j \lambda_j^k, \quad k = 0, 1, 2, \ldots. \tag{4.4}$$

It is easy to see from (4.2) that the number of poles of the function L equals the number of *distinct* zeros of the polynomial p. But the Cauchy index $\mathrm{Ind}_{-\infty}^{+\infty}(L)$ equals the number of *distinct real* zeros of p, since all poles of L are simple and since the residue of L at each pole is positive, [61, 36]. Also from (4.2)–(4.3) and from Theorem 1.2 it follows that the rank of the matrix $S = \|s_{i+j}\|_0^\infty$ consisting of the Newton sums (4.4) is finite and is equal to $m\,(\leq n)$, the number of distinct zeros of p.

As before, we denote by $D_j(L)$ $(j = 1, 2, \ldots, m)$ the leading principal minors of the matrix $S(L) = \|s_{i+j}\|_0^\infty$ (see (1.6)). Then from Theorem 2.11 we obtain

$$\mathrm{Ind}_{-\infty}^{+\infty}(L) = m - 2\,\mathrm{V}(1, D_1(L), D_2(L), \ldots, D_m(L)). \tag{4.5}$$

The above facts and the formula (4.5) imply the following theorem.

Theorem 4.1 ([36, 61])**.** *The number of distinct pairs of non-real zeros of the polynomial p equals $\mathrm{V}(1, D_1(L), D_2(L), \ldots, D_m(L))$.*

Corollary 4.2.

$$\mathrm{V}(1, D_1(L), D_2(L), \ldots, D_m(L)) \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

We also consider the determinants $\widehat{D}_j(L)$ $(j = 1, 2, \ldots, m)$ defined by (1.9) and introduce the following notation for counting zeros:

Definition 4.3. Let r denote the number of *distinct real* zeros of the polynomial p and let r^+ and r^- be the numbers of *distinct positive* and *negative* zeros of p, respectively.

Theorem 2.14 implies the following simple fact.

Theorem 4.4. *Let the numbers k and l be defined as follows*[27]*:*

$$k = \mathrm{V}(1, D_1(L), D_2(L), \ldots, D_m(L)), \qquad l = \mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \ldots, \widehat{D}_m(L)),$$

Then the number of distinct pairs of non-real zeros of the polynomial p equals k and

$$\begin{aligned} r &= m - 2k; \\ r^- &= l - k; \\ r^+ &= \begin{cases} m - k - l & \text{if } p(0) \neq 0, \\ m - k - l - 1 & \text{if } p(0) = 0. \end{cases} \end{aligned} \tag{4.6}$$

Proof. By Theorem 4.4, k is the number of distinct pairs of non-real zeros of the polynomial p. Since $\mathrm{Ind}_{-\infty}^{+\infty}(L) = r$ as mentioned above, we conclude

$$r = m - 2k. \tag{4.7}$$

Now we observe that

$$r^+ = \mathrm{Ind}_0^{+\infty}(L), \qquad r^- = \mathrm{Ind}_{-\infty}^0(L). \tag{4.8}$$

If $p(0) \neq 0$, then $r = r^+ + r^-$ and $|L(0)| > 0$. So, from (4.8) and (2.37)–(2.38) we obtain

$$r^- = l - k, \quad r^+ = m - k - l.$$

[27] Recall that the numbers $\mathrm{V}(1, D_1(L), D_2(L), \ldots, D_m(L))$ and $\mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \ldots, \widehat{D}_m(L))$ must be calculated according to Frobenius Rule 2.4.

Now let $p(0)=0$. Then
$$r = r^+ + r^- + 1 = m - 2k, \tag{4.9}$$
In this case, the number σ_1 equals 1 in the formulæ (2.39)–(2.40) since all the residues of the function L are positive (see (4.2)), whereas the number σ_2 equals σ_1 since all poles of L are simple. Thus, from (4.8) and (2.39)–(2.40) we obtain $r^- = l - k$, $\quad r^+ = m - k - l - 1$, as required. ∎

Corollary 4.5.
$$\mathrm{V}(1, D_1(L), D_2(L), \dots, D_m(L)) \leq \mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \dots, \widehat{D}_m(L)).$$

Remark 4.6. Since $s_0 = n$, we have $D_1(L) = s_0 = n > 0$. This fact means that the polynomial p has at least one zero. If $D_j(L) = 0$ for $j \geq 2$, then p has exactly one zero of multiplicity n.

Our next statement is a slight modification (we use another continued fraction) and generalization (we cover the case $p(0) = 0$) of Theorem 3.5 from [65] (see also [84]). However, [65] uses different methods.

Theorem 4.7. *Let the polynomial p be defined by* (4.1). *Then its logarithmic derivative L has a Stieltjes continued fraction expansion*
$$L(z) = \cfrac{1}{c_1 z + \cfrac{1}{c_2 + \cfrac{1}{c_3 z + \cfrac{1}{\ddots + \cfrac{1}{T}}}}}, \quad c_j \in \mathbb{R}, \quad c_j \neq 0, \tag{4.10}$$
where
$$T = \begin{cases} c_{2m} & \text{if } p(0) \neq 0, \\ c_{2m-1} z & \text{if } p(0) = 0 \end{cases} \tag{4.11}$$
if and only if L satisfies the inequalities
$$\begin{aligned} D_j(L) &\neq 0, & j &= 1, 2, \dots, m, \\ \widehat{D}_j(L) &\neq 0, & j &= 1, 2, \dots, m-1. \\ D_j(L) = \widehat{D}_j(L) &= 0, & j &> m. \end{aligned}$$
where $m \leq \deg p$.

In that case, p has exactly m distinct zeros. Moreover, if the number of negative coefficients c_{2j-1} equals k, and the number of positive coefficients c_{2j} equals l, then the number of distinct pairs of nonreal zeros of p and the number of its distinct real, positive and negative zeros are given by the formulæ (4.6).

Proof. The theorem follows immediately from Theorems 1.46, 2.15 and 4.4. ∎

From (1.114)–(1.115) it follows that the coefficients c_i in (4.10)–(4.11) can be found as follows:
$$c_{2j-1} = \frac{\widehat{D}^2_{j-1}(L)}{D_{j-1}(L) \cdot D_j(L)}, \quad j = 1, 2, \dots, m. \tag{4.12}$$
$$c_{2j} = -\frac{D^2_j(L)}{\widehat{D}_{j-1}(L) \cdot \widehat{D}_j(L)}, \quad j = 1, 2, \dots, \widetilde{m}, \tag{4.13}$$
where $\widetilde{m} = m$ if $p(0) \neq 0$, $\widetilde{m} = m - 1$ if $p(0) = 0$, and $D_0(L) \equiv 1$, $\widehat{D}_0(L) \equiv 1$.

Thus, Theorem 4.4 expresses the numbers of positive, negative and non-real zeros in terms of the number of sign changes in the sequences of the minors $D_j(L)$ and $\widehat{D}_j(L)$, and Theorem 4.7 does the same in terms of the Stieltjes continued fraction of L, provided that L has such a continued fraction expansion. Now we will obtain formulæ for those numbers in terms of the coefficients a_j of the given polynomial p.

Consider the following $2n \times 2n$ matrix.

$$\mathcal{D}_{2n}(p) := \begin{pmatrix} na_0 & (n-1)a_1 & (n-2)a_2 & \dots & a_{n-1} & 0 & \dots & 0 & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & \dots & 0 & 0 \\ 0 & na_0 & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & \dots & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & \dots & 0 & 0 \\ 0 & 0 & na_0 & \dots & 3a_{n-3} & 2a_{n-2} & \dots & 0 & 0 \\ 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & na_0 & (n-1)a_1 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}, \tag{4.14}$$

which is a Hurwitz-type matrix constructed with the coefficients of the polynomials p and with the coefficients of its derivative p'. According to Definition 1.42, the matrix $\mathcal{D}_{2n}(p)$ is $\mathcal{H}_{2n}(p,q)$ (see (1.125)), where $q = p'$. Denote the leading principal minors if the matrix $\mathcal{D}_{2n}(p)$ by $\delta_j(p)$, $j = 1, 2, \dots, 2n$. We remind the reader that $\delta_{2n-1}(p) = a_0\mathbf{D}(p)$, where $\mathbf{D}(p)$ is the discriminant of p (see (1.29)).

Theorem 4.8. *The polynomial p has exactly $m \leq n$ distinct zeros if and only if*

$$\delta_{2m-1}(p) \neq 0, \quad \text{and} \quad \delta_j(p) = 0 \quad \text{for} \quad j > 2m. \tag{4.15}$$

At the same time, if

$$k = \mathrm{V}(1, \delta_1(p), \delta_3(p), \dots, \delta_{2m-1}(p)), \qquad l = \mathrm{V}(1, \delta_2(p), \delta_4(p), \dots, \delta_{2m}(p)),$$

then the number of distinct pairs of non-real roots of the polynomial p equals k and

$$\begin{aligned} r &= m - 2k\,; \\ r^+ &= l - k\,; \\ r^- &= \begin{cases} m - k - l & \text{if} \quad p(0) \neq 0, \\ m - k - l - 1 & \text{if} \quad p(0) = 0, \end{cases} \end{aligned}$$

where r is the number of distinct real roots of p, and r^- and r^+ are the numbers of distinct negative and positive roots of p, respectively.

Proof. By Definition 1.42, we have

$$\delta_j(p) = \Delta_j(p, p'), \quad j = 1, 2, \dots$$

Therefore, from Theorem 1.45 (see (1.136)–(1.137)) we obtain

$$\delta_{2j-1}(p) = a_0^{2j-1} D_j(L), \quad j = 1, 2, \dots, n; \tag{4.16}$$

$$\delta_{2j}(p) = (-1)^j a_0^{2j} \widehat{D}_j(L), \quad j = 1, 2, \dots, n. \tag{4.17}$$

Then, from Theorem 1.2 and (4.16) it follows that the rank of the matrix $S = \|s_{j+k}\|_0^\infty$ equals m if and only if the condition (4.15) holds. At the same time, we have $\widehat{D}_{m-1}(L) \neq 0$, $\widehat{D}_m(L) = 0$ if and only if $p(0) = 0$, and $\widehat{D}_m(L) \neq 0$ if and only if $p(0) \neq 0$, and $\widehat{D}_j(L) = 0$ for $j > m$.

It suffices to note that (4.16) implies that

$$k = \mathrm{V}(1, \delta_1(p), \delta_3(p), \dots, \delta_{2m-1}(p)) = \mathrm{V}(1, D_1(L), D_2(L), \dots, D_m(L)) \tag{4.18}$$

since $a_0 > 0$. But from (4.17) we have

$$\begin{aligned} l \;=\; & \mathrm{V}(1, \delta_2(p), \delta_4(p), \dots, \delta_{2m}(p)) = \mathrm{V}(1, -\widehat{D}_1(L), \widehat{D}_2(L), \dots, (-1)^m \widehat{D}_m(L)) \\ & = m - \mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \dots, \widehat{D}_m(L)), \end{aligned} \tag{4.19}$$

which gives $\mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \dots, \widehat{D}_m(L)) = m - l$ if $\widehat{D}_m(L) \neq 0$, and if[28] $\widehat{D}_m(L) = 0$, then $\mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \dots, \widehat{D}_m(L)) = m - 1 - l$. Now the assertion of the theorem follows from (4.18), (4.19) and from Theorem 4.4. □

[28] In this case, $\widehat{D}_{m-1}(L) \neq 0$, according to Corollary 1.4.

Remark 4.9. For the polynomial p we have $\delta_1(p) = a_0 D_1(L) = a_0 n > 0$.

Using the formulæ (4.16)–(4.17), we can represent the coefficients c_i of the Stieltjes continued fraction expansion (4.10)–(4.11) of the function L in terms of the determinants $\delta_i(p)$:

$$c_i = \frac{\delta_{i-1}^2(p)}{\delta_{i-2}(p)\cdot\delta_i(p)}, \quad i = 1, 2, \dots, 2m, \tag{4.20}$$

where $\delta_{-1}(p) \equiv \dfrac{1}{a_0}$, $\delta_0(p) \equiv 1$.

4.2 Polynomials with real zeros

We now provide several explicit criteria for a polynomial to have only real zeros. These criteria, just as those developed before, use the Hankel and Hurwitz minors made of the coefficients of a given polynomial.

Let us again consider the polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n, \qquad a_1, \dots, a_n \in \mathbb{R},\ a_0 > 0. \tag{4.21}$$

and let

$$L(z) = \frac{p'(z)}{p(z)} = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots \tag{4.22}$$

be its logarithmic derivative. Theorem 4.4 directly implies the following results.

Theorem 4.10. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are real, if and only if*

$$\begin{aligned} &D_j(L) > 0, \qquad j = 1, 2, \dots, m,\\ &D_j(L) = 0, \qquad j > m. \end{aligned}$$

Theorem 4.11. *Let the polynomial p have exactly $m\,(\leq n)$ distinct zeros all of which are real. Then the number of distinct negative zeros of the polynomial p equals* $\mathrm{V}(1, \widehat{D}_1(L), \widehat{D}_2(L), \dots, \widehat{D}_m(L))$. *Moreover,*

$$\widehat{D}_j(L) = 0, \qquad j > m.$$

and

$$\begin{cases} \widehat{D}_m(L) \neq 0 & \textit{if} \quad p(0) \neq 0,\\ \widehat{D}_m(L) = 0 \ \textit{ and } \ \widehat{D}_{m-1}(L) \neq 0 & \textit{if} \quad p(0) = 0. \end{cases}$$

Proof. The polynomial p takes value 0 at 0 if and only if the function L has a pole at zero. Therefore, the assertion of the theorem follows from Corollary 1.4 and Theorem 4.4. ∎

Remark 4.12. According to Theorem 3.32, if the polynomial p has only real zeros, then the number of its positive zeros equals $\mathrm{V}^{-}(a_0, a_1, a_2, \dots, a_n)$, i.e., the number of strong sign changes in the sequence of the coefficients of the polynomial p. Obviously, the number of its negative zeros is equal to $\mathrm{P}^{-}(a_0, a_1, a_2, \dots, a_n)$, i.e., the number of strong sign retentions in the sequence of its coefficients.

Also from Theorem 4.7 we obtain the following simple corollary.

Corollary 4.13. *Let the polynomial p have exactly $m\,(\leq n)$ distinct zeros, all of which are real, and let its logarithmic derivative L have the Stieltjes continued fraction expansion* (4.10)–(4.11). *Then*

$$c_{2j-1} > 0, \qquad j = 1, 2, \dots, m, \tag{4.23}$$

and the number of positive coefficients c_{2j} equals the number of negative distinct zeros of the polynomial p. The coefficients c_i can be found by the formulæ (4.12)–(4.13) *or* (4.20).

Let us consider the Hankel matrix

$$S(L) = \|s_{i+j}\|_0^\infty, \tag{4.24}$$

made of the coefficients of the series (4.22).

From Theorems 3.4, 3.18, 4.10, 4.11, Corollary 3.19 and Remark 4.12 we obtain the following straightforward consequences, the first two addressing the case when all zeros of p are positive, and the next two when all zeros are negative.

Corollary 4.14. *The polynomial p has only positive zeros and exactly $m\,(\leq n)$ of them are distinct if and only if the matrix $S(L)$ defined by* (4.24) *is strictly totally positive of rank m.*

Corollary 4.15. *The polynomial p has only positive zeros and exactly $m\,(\leq n)$ of them are distinct if and only if the matrix $S(L)$ is positive definite of rank m and $a_{j-1}a_j < 0$ $(j = 1, 2, \ldots, n)$.*

Corollary 4.16. *The polynomial p has only negative zeros and exactly $m\,(\leq n)$ of them are distinct if and only if the matrix $S(L)$ is strictly sign regular of rank m.*

Corollary 4.17. *Let all the coefficients of the polynomial p be of the same sign. Then all zeros of the polynomial p are negative and exactly $m\,(\leq n)$ of them are distinct if and only if the matrix $S(L)$ is positive definite of rank m.*

If $m = n$ in Theorems 4.10–4.11 and Corollaries 4.13–4.17, then we obtain criteria of reality (negativity, positivity) and *simplicity* for all zeros of a given polynomial.

Now we give criteria of reality for all zeros of a given polynomial in terms of the determinants $\delta_j(p)$, which are the leading principal minors of the matrix $\mathcal{D}_{2n}(p)$ defined by (4.14) (see e.g. [56, 102]).

Theorem 4.18. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are real, if and only if*

$$\delta_1(p) > 0,\ \delta_3(p) > 0, \ldots,\ \delta_{2m-1}(p) > 0. \tag{4.25}$$

$$\delta_j(p) = 0, \quad j > 2m, \tag{4.26}$$

Theorem 4.19. *Let the polynomial p have exactly $m\,(\leq n)$ distinct zeros, all of which are real. If*

$$l = \mathrm{V}(1, \delta_2(p), \delta_4(p), \ldots, \delta_{2m}(p)),$$

then the number of distinct positive zeros of the polynomial p equals l (when $p(0) \neq 0$) or $l-1$ (when $p(0) = 0$). The number of all positive zeros of p, counting multiplicities, is equal to $\mathrm{V}^-(a_0, a_1, a_2, \ldots, a_n)$.

Proof. The theorem follows from Theorem 4.8, formulæ (4.16)–(4.17) and Remark 4.12. □

From Theorems 4.19 and 4.18 and our previous results we obtain the following evident consequences:

Corollary 4.20. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are positive, if and only if* (4.25)–(4.26) *hold and*

$$\delta_{2j-2}(p)\delta_{2j}(p) < 0, \quad j = 1, 2, \ldots, m, \quad (\delta_0(p) := 1).$$

Corollary 4.21. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are positive, if and only if* (4.25)–(4.26) *hold and*

$$a_{j-1}a_j < 0, \quad j = 1, 2, \ldots, n.$$

Corollary 4.22. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are negative, if and only if the equalities* (4.26) *hold and*

$$\delta_1(p) > 0,\ \delta_2(p) > 0, \ldots,\ \delta_{2m-1}(p) > 0,\ \delta_{2m}(p) > 0.$$

Corollary 4.23. *Let all the coefficients of the polynomial p be positive. The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are negative, if and only if* (4.25)–(4.26) *hold.*

For $m = n$, Theorems 4.18–4.19 and Corollaries 4.20–4.23 become criteria of reality (negativity, positivity) and *simplicity* for all zeros of a given polynomial.

Corollary 4.22 with $m = n$ is *per se* the Hurwitz stability criterion for polynomials whose zeros are real and simple; it is analogous to the standard Hurwitz criterion of polynomial stability [36, Chapter XV, Section 6]. In turn, Corollary 4.23 with $m = n$ is an analogue of the Liénard and Chipart criterion [36, Chapter XV, Section 13]. Just as the criterion of Liénard and Chipart has a number of versions [36, Chapter XV, Section 13], we can also obtain other versions of Corollary 4.23, e.g., as follows:

Corollary 4.24. *Let all the coefficients of the polynomial p be positive. Then all roots of the polynomial p are negative and distinct if and only if the following inequalities hold:*

$$\delta_2(p) > 0,\ \delta_4(p) > 0, \dots,\ \delta_{2n}(p) > 0. \tag{4.27}$$

Proof. The theorem follows from Theorem 3.34 applied to the pair p, p'. □

Note that all criteria of Hurwitz stability require n inequalities on the coefficients of a polynomial of degree n [11, 36], while the criteria of simplicity and negativity of zeros require $2n$ inequalities on the coefficients of a polynomial of degree n.

The similarity between Hurwitz stable polynomials and polynomials with simple and negative zeros is also displayed by the following theorem, which is an analogue of the famous fact that the Hurwitz matrix of a Hurwitz stable polynomial is totally nonnegative [6, 57, 46, 55].

Theorem 4.25. *The polynomial p of degree n has only nonpositive zeros if and only if its matrix $\mathcal{D}_{2n}(p)$ defined in* (4.14) *is totally nonnegative.*

Proof. If $n = 0$, then the assertion is evident. Let $n \geq 1$. From Theorems 4.10–4.11 and 3.4 it follows that the polynomial p has nonpositive zeros if and only if its logarithmic derivative is an R-function of negative type with nonpositive poles and negative roots. Since $\mathcal{D}_{2n}(p) = \mathcal{H}_{2n}(p, p')$, the necessity direction of the theorem follows from Theorem 3.46 applied to the pair (p, p'). The sufficiency can be proved as in Theorem 3.44 using the factorization (3.72) where $\mathcal{T}_k(g)$ is totally nonnegative whenever $g = \gcd(p, p')$. □

From Theorem 3.44 we derive the following corollary.

Corollary 4.26. *The polynomial p of degree n has only nonpositive zeros if and only if the infinite matrix $H(p, p')$ defined in* (1.123) *is totally nonnegative.*

Now, the following corollaries about Stieltjes continued fractions of logarithmic derivatives can be derived from Theorems 3.4, 4.10 and 4.11 and from Corollary 4.13:

Corollary 4.27. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are positive, if and only if its logarithmic derivative L has a Stieltjes continued fraction expansion* (4.10)–(4.11) *where the inequalities* (4.23) *hold and where*

$$c_{2j} < 0, \quad j = 1, 2, \dots, m.$$

Corollary 4.28. *The polynomial p has exactly $m\,(\leq n)$ distinct zeros, all of which are negative, if and only if its logarithmic derivative L has a Stieltjes continued fraction expansion* (4.10)–(4.11) *where*

$$c_i > 0, \qquad i = 1, 2, \dots, 2m.$$

Note that, in Corollaries 4.27–4.28, all zeros of the polynomial p are simple if and only if $m = n$.

At last, we present the following fact, which is a simple consequence of Corollary 3.52.

Theorem 4.29. *The polynomial*

$$g(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_0 \neq 0, \quad a_n > 0 \tag{4.28}$$

has all negative zeros if and only if the infinite matrix

$$\mathcal{D}_\infty(g) := \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \dots \\ 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 & 6a_6 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.29}$$

is totally nonnegative.

Proof. In fact, consider the logarithmic derivative L of the polynomial g

$$L(z) = \frac{g'(z)}{g(z)} = \frac{a_1 + 2a_2 z + \cdots + n a_n z^{n-1}}{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots$$

Suppose that the polynomial g has exactly $m\,(\leq n)$ distinct zeros, all of which are negative. Then by Theorems 1.2, 4.10 and 4.11, we have

$$D_j(L) > 0, \qquad j = 1, 2, \ldots, m, \tag{4.30}$$

$$(-1)^j \widehat{D}_j(L) > 0, \qquad j = 1, 2, \ldots, m, \tag{4.31}$$

$$D_j(L) = \widehat{D}_j(L) = 0, \qquad j > m. \tag{4.32}$$

According to Theorems 3.4 and 3.9, from (4.30)–(4.32) we obtain that the logarithmic derivative L of the polynomial g is an R-function of negative type with all poles and zeros negative. Moreover, all common zeros of the numerator and the denominator of the function L are also negative. The converse statement is obviously true.

If now we take $h = g'$ in Corollary 3.52, then $R = L$ and $H_\infty(g, g') = \mathcal{D}_\infty(g)$. So, the assertion of the theorem immediately follows from Corollary 3.52. □

Remark 4.30. The necessary condition in this theorem is essentially known (see [28, Remark 3.23]) but the sufficient condition is most likely new.

For the polynomial (4.28) with negative zeros, the total nonnegativity of the matrix (4.29) implies two properties of independent interest (cf. [28, p. 66]):

$$a_j^2 - a_{j-1}a_{j+1} = \begin{vmatrix} a_j & a_{j+1} \\ a_{j-1} & a_j \end{vmatrix} \geq 0, \quad j = 1, 2, \ldots, n-1 \qquad \text{(log-concavity)}$$

and

$$a_j^2 - \frac{j+1}{j} \cdot a_{j-1}a_{j+1} = \frac{1}{j} \cdot \begin{vmatrix} j a_j & (j+1)a_{j+1} \\ a_{j-1} & a_j \end{vmatrix} \geq 0, \quad j = 1, 2, \ldots, n-1 \qquad \text{(weak Newton's inequalities)}$$

Remark 4.31. From Theorem 4.29 one can also prove Corollary 3.56.

At last, we present a couple of concrete examples.

Example 4.32. *Consider the following real polynomial*

$$f(z) = z^3 + az + b, \quad a, b \in \mathbb{R}.$$

By our methods, we obtain a well-known fact [64] that the polynomial f has only real zeros if and only if the coefficients a and b satisfy the inequality[29]

$$4a^3 + 27b^2 \leq 0, \tag{4.33}$$

At the same time, we show that the zeros of f cannot all be of the same sign.

In fact, construct the matrix $\mathcal{A}_6(f)$ corresponding to the polynomial f

$$\mathcal{D}_6(f) = \begin{pmatrix} 3 & 0 & a & 0 & 0 & 0 \\ 1 & 0 & a & b & 0 & 0 \\ 0 & 3 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & a & b & 0 \\ 0 & 0 & 3 & 0 & a & 0 \\ 0 & 0 & 1 & 0 & a & b \end{pmatrix}$$

and find its leading principal minors $\delta_j(f)$ $(j = 1, 2, \ldots, 6)$:

$$\delta_1(f) = 3; \ \ \delta_2(f) = \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0; \ \ \delta_3(f) = \begin{vmatrix} 3 & 0 & a \\ 1 & 0 & a \\ 0 & 3 & 0 \end{vmatrix} = -6a; \tag{4.34}$$

[29] We note that $4a^3 + 27b^2$ is the discriminant of the polynomial f.

$$\delta_4(f) = \begin{vmatrix} 3 & 0 & a & 0 \\ 1 & 0 & a & b \\ 0 & 3 & 0 & a \\ 0 & 1 & 0 & a \end{vmatrix} = -4a^2; \quad \delta_5(f) = \begin{vmatrix} 3 & 0 & a & 0 & 0 \\ 1 & 0 & a & b & 0 \\ 0 & 3 & 0 & a & 0 \\ 0 & 1 & 0 & a & b \\ 0 & 0 & 3 & 0 & a \end{vmatrix} = -4a^3 - 27b^2;$$

$$\delta_6(f) = |\mathcal{D}_6(f)| = b\delta_5(f).$$

According to Theorem 4.18, the polynomial f has only real zeros if and only if the following inequalities hold:

$$\delta_1(f) > 0, \quad \delta_3(f) \geq 0, \quad \delta_5(f) \geq 0. \tag{4.35}$$

Moreover, the equality $\delta_3(f) = 0$ must imply $\delta_5(f) = 0$. But it is easy to see that the case $\delta_3(f) = \delta_5(f) = 0$ is possible if and only if $a = b = 0$.

The first inequality (4.35) *holds automatically (see Remark 4.9). The second inequality* (4.35) *gives $a \leq 0$. From the third inequality it follows that the necessary and sufficient condition for the polynomial f to have only real zeros is the inequality* (4.33)*. This inequality also holds for $a = b = 0$ and implies the inequality $a \leq 0$.*

If the polynomial f has only real zeros, then, according to Corollaries 4.22–4.23, all its zeros are negative or positive if all its coefficients are nonzero. But the coefficient of $f(z)$ at z^2 vanishes, therefore, f cannot have all zeros of the same sign.

Thus, we proved that f has only real zeros if and only if a and b satisfy the inequality (4.33)*. We also proved that f cannot have only positive or only negative zeros for any real a and b.*

Example 4.33. *From Theorem 4.18 it follows that if the polynomial p defined by* (4.21) *is of degree $n \geq 3$ with $a_1 = a_2 = 0$ and $a_3 \neq 0$, then p cannot have only real zeros since we have*

$$\delta_3(p) = 0,$$

and

$$\delta_5(p) = -9na_0^3a_3^2 < 0.$$

Therefore,

$$\operatorname{sign} \delta_5(p) = -\operatorname{sign} a_0 = -\operatorname{sign} \delta_1(p).$$

This contradicts the inequalities (4.25).

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